Theory for Full Availability Group,

Delay System

(Exercises included)

From TETRAPRO, edited by Mr. H. Leijon, ITU

UNION INTERNATIONALE DES TELECOMMUNICATIONS
INTERNATIONAL TELECOMMUNICATION UNION
UNION INTERNACIONAL DE TELECOMUNICACIONES

Basic Teletraffic Theory (T)

THEORY FOR FULL AVAILABILITY GROUP, DELAY SYSTEM (TFD)

Contents

1. Assumptions

A delay system is characterised by the fact that blocked calls can wait and be served later when a device becomes free.

For the assumption (TGD 2.9) this means that

$$W(p) = 1 \text{ for all } p \qquad \text{(TFD 1.1)}$$

It also means that a delay system must be able to accept as many queuing calls as can possibly be produced. A system, which can accept only a limited number of waiting calls, functions above that limit as a loss system and should be regarded as a combined delay and loss system.

Here we shall deal only with genuine delay systems where $W(p) = 1$ for all p.

One may either assume that all queuing calls wait until served or that waiting calls can give up. Both of these assumptions are dealt with in the sequel. The former leads to simpler mathematical expressions, the latter to improved realism.

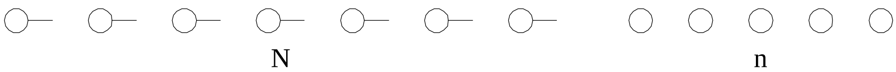
For the complete treatment of a delay system it is necessary to define the queue discipline, i.e. the way in which the next served call is selected from the queue. One may distinguish between essentially three methods of handling queues:

1. ordered queue (first come, first served);
2. random queue (every queuing call has the same probability of being the next one to be served;
3. priority queue (all queuing calls have different priorities, that with the highest priority is served first).

There are also combinations of these queue disciplines, as well as cases in which two-step queuing is used (from an outer room a number is taken into an inner room; all in the inner room are served before new calls are admitted).

The queue discipline has an influence on the waiting time distribution, but not generally on the mean waiting time.

Consider a full availability group *(N,n)*



The state of the system is defined as *(p)*, where

$$0 \le p \le N \qquad \text{(TFD 1.2)}$$

For $p > n$ the state *(p)* implies that n devices are engaged and that $p-n$ calls are waiting.

It is evident that N must be $> n$ for a delay to arise. This excludes the assumption for the Bernoulli distribution being considered for a waiting system. For the case $N = \infty$, $n = \infty$, it is not considered that a delay can arise. Therefore, the assumptions for the Poisson distribution are not valid here.

Thus the delay systems are limited to cases when

$$N > n \qquad \text{(TFD 1.3)}$$

where N can be finite or infinite, while n must always be finite.

For the intensities of termination it is assumed that

$$\left.\begin{aligned} \mu_p &= \frac{p}{s} && \text{for } 0 \le p \le N \\ \mu_p &= \frac{n}{s} + \frac{\Theta}{s} \cdot (p-n) && \text{for } \quad p > n \end{aligned}\right\} \qquad \text{(TFD 1.4)}$$

$\Theta > 0$ means that a waiting call gives up with the intensity Θ/s.
$\Theta = 0$ means that all calls wait until they are served. The assumption
$\Theta > 0$ implies that a waiting call continues to wait for a time described by the exponential distribution

$$f(t) = \frac{\Theta}{s} \cdot e^{-\frac{\Theta}{s} \cdot t} \qquad \text{(TFD 1.5)}$$

with the mean value $\frac{s}{\Theta}$

Expression (TFD 1.4) gives the following general solution:

$$\left.\begin{aligned} [p] &= \frac{s^p \cdot \prod_{\nu=0}^{p-1} y(\nu)}{p!} \cdot [0] && \text{for } 0 \le p \le N \\ [p] &= \frac{s^p}{\Theta^{p-n}} \cdot \frac{\prod_{\nu=0}^{p-1} y(\nu)}{n! \prod_{\nu=n+1}^{p} \left(n \cdot \frac{1-\Theta}{\Theta} + \nu \right)} \cdot [0] && \text{for } n < p \le N \end{aligned}\right\} \qquad \text{(TFD 1.4a)}$$

where N may either be finite or infinite.

For the intensities of increase it is assumed that

$$\lambda = y(p) \cdot W(p) \qquad \text{(TFD 2.9)}$$

where $W(p) = 1$ for all p. $\quad 0 \le p \le N$

and

$$\left.\begin{aligned} y(p) &= (N-p) \cdot \beta && \textbf{EB} \ \ type \\ y(p) &= y && \textbf{E} \ \ type \\ y(p) &= a \cdot (\gamma + p) && \textbf{TNB} \ type \end{aligned}\right\} \qquad \text{(TFD 1.6)}$$

Insertion of (TFD 1.4) and (TGD 2.9) in (TGD 1.11) gives

$$\left.\begin{aligned} p \cdot [p] &= s \cdot y(p-1) \cdot [p-1] && \text{for } 0 \le p \le n \\ (n + \Theta \cdot (p-n)) \cdot [p] &= s \cdot y(p-1) \cdot [p-1] && \text{for } n < p \le N \end{aligned}\right\} \qquad \text{(TFD 1.7)}$$

From (TFD 1.7) the solution for different assumptions regarding y (p) as given in (TFD 1.6) is then obtained by recursions and using

$$\sum_{p=0}^{N} [p] = 1$$

it is understood that increased Θ means that the waiting will become shorter before the calls give up. If $\Theta = 1$, no call will wait at all and the system becomes a loss system.

2. Some characteristics of delay systems

In addition to the quantities defined for loss systems, the waiting times must also be specified for delay systems.

Time congestion

Time congestion is here defined as the probability that all devices are engaged. Consequently,

$$E = \sum_{p=n}^{N} [\,p\,] \qquad \text{(TFD 2.1)}$$

Call congestion

Call congestion is defined as the probability that a call must wait = the proportion of calls which must wait. Consequently,

$$B = P(>0) = \frac{\sum_{p=n}^{N} y(p)\cdot[\,p\,]}{\sum_{p=0}^{N} y(p)\cdot[\,p\,]} \qquad \text{(TFD 2.2)}$$

Probability of giving up waiting

The expected proportion of calls that give up waiting is calculated as

$$B_{gu} = \frac{\sum_{p=n+1}^{N} [\,p\,]\cdot\frac{\Theta}{s}\cdot(n-p)}{\sum_{p=0}^{N} y(p)\cdot[\,p\,]} \qquad \text{(TFD 2.2a)}$$

where N may be either finite or infinite.

Carried and offered traffic

The traffic carried by the n devices is

$$A^1 = \sum_{p=0}^{n\text{-}1} p[\,p\,] + \sum_{p=n}^{N} n\cdot[\,p\,] \qquad \text{(TFD 2.3)}$$

The traffic offered is

$$A = \sum_{p=0}^{N} s\cdot y(p)\cdot[\,p\,] \qquad \text{(TFD 2.4)}$$

The relation between A and A^1 is obtained from

$$\left.\begin{aligned} s\cdot y(p)\cdot[\,p\,] &= (p+1)\cdot[\,p+1\,] && \text{for } 0 \le p < n \\ s\cdot y(p)\cdot[\,p\,] &= (n+\Theta\cdot(p-n+1))\cdot[\,p+1\,] && \text{for } n \le p < N \end{aligned}\right\} \qquad \text{(TFD 1.7)}$$

observing that $y(N) = 0$. When there are no free sources, no more calls can be offered.

$$\sum_{p=0}^{N} s\cdot y(p)\cdot[\,p\,] = \sum_{p=1}^{n} p\cdot[\,p\,] + \sum_{p=n+1}^{N} \big(n+\Theta\cdot(p-n)\big)\cdot[\,p\,]$$

Consequently,

$$A = A^1 + \Theta \cdot Q \qquad \text{(TFD 2.5)}$$

where

$$Q = \sum_{p=n+1}^{N} (p-n) \cdot [p] \qquad \text{(TFD 2.6)}$$

Q is the mean number of waiting calls in the system = mean length of queue, calculated over the entire period (including occasions when there are no waiting calls).

It is evident that

$$\boxed{\begin{array}{l} A = A^1 \ \text{ for } \ \Theta = 0 \\ A > A^1 \ \text{ for } \ \Theta > 0 \end{array}} \qquad \text{(TFD 2.7)}$$

The improvement factor

The improvement factor is defined as the additional amount of traffic that can be carried if the number of devices is increased from n to $n + \Delta n$.

$$F = A^1(n+\Delta n) - A^1(n) \qquad \text{(TFD 3.10)}$$

Probability that x specific devices are engaged

For a delay system according to (TGD 3.11) this probability is

$$H_x = \sum_{p=x}^{n-1} [p] \cdot \frac{\binom{n-x}{p-x}}{\binom{n}{p}} + \sum_{p=n}^{N} [p] \qquad \text{(TFD 3.11)}$$

Waiting times

The waiting time distribution depends on the queue discipline. If the frequency function of the waiting times is denoted by *f (t)*, the mean waiting time for the waiting calls is

$$u = \int_{t=0}^{\infty} t \cdot f(t)dt \qquad \text{(TFD 2.8)}$$

while the mean waiting time for all calls is

$$U = P(>0) \cdot \int_{t=0}^{\infty} t \cdot f(t)dt \qquad \text{(TFD 2.9)}$$

P (>0) = the probability that a call must wait, i.e.

$$U = P(>0) \cdot u \qquad \text{(TFD 2.9a)}$$

$$U < u \qquad \text{(TFD 2.10)}$$

Waiting time distributions

The probability of long (troublesome) waiting times is generally most important from the service point of view. The probability that a waiting call has to wait longer than a time t_0:

$$P_V(>t_0) = P_V(t>t_0) = \int_{x=t_0}^{\infty} f(x)dx \qquad \text{(TFD 2.11)}$$

The probability that an arbitrary call has to wait longer than a time t_0:

$$P\ (>t_0) = P\ (>0)\cdot P_V(>t_0) \qquad \text{(TFD 2.12)}$$

$$P\ (>t_0) = P\ (>0)\cdot \int_{x=t_0}^{\infty} f(x)dx \qquad \text{(TFD 2.12a)}$$

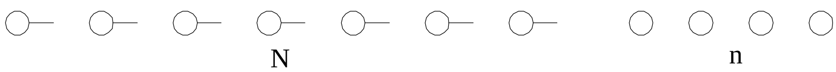
The waiting time distribution is depending on the queue discipline, while the mean waiting time is nearly always independent of the queue discipline.	(TFD 2.13)

Since waiting can occur only when the number of sources is greater than the number of devices $(N > n)$, there are only three distributions to be considered here, namely those of the types EB, E and TNB. For each of these distributions there are two possible cases corresponding to whether waiting calls are assumed to give up or not $(\Theta > 0 \text{ or } \Theta = 0)$. The assumption $\Theta > 0$ increases the realism of the models but increases also the complexity of the formulas.

Since the distribution of the TNB type has not yet been used in theory, there only remain EB and E to be described in more detail.

For the loss systems (TFL) we investigated one traffic parameter at a time, for all the different basic cases (EB, E etc.). Here we found it suitable to do quite the opposite, namely to investigate one basic case at a time, for all the different traffic parameters.

3. Limited number of sources: delay system of Engset type



Assumption (TFD 1.3) + (TFD 1.4) + (TFD 1.6EB) gives

$$\left.\begin{array}{ll}
0 \le p \le N & \\
N > n & \\
\mu_p = \dfrac{p}{s} & for\ \ 0 \le p \le n \\
\mu_p = \dfrac{n}{s} + \dfrac{\Theta}{s}\cdot(p-n) & for\ \ n < p \le N \\
\lambda_p = y(p)\cdot W(p) & \\
y(p) = (N-p)\cdot\beta & \\
W(p) = 1 & for\ \ 0 \le p \le N
\end{array}\right\} \qquad \text{(TFD 3.1)}$$

$\Theta = 0$: all calls wait until served

$\Theta > 0$: waiting calls give up waiting with the intensity $\dfrac{\Theta}{s}$

From

$$\lambda_{p-1} \cdot [\,p-1\,] = \mu_p \cdot [\,p\,] \qquad \text{(TGD 1.11)}$$

we obtain for the assumption (TFD 1.4) + (TFD 1.6EB):

$$\boxed{\begin{array}{ll}
[p] = \binom{N}{p} \cdot \alpha^p \cdot [0] & 0 \le \mathrm{p} \le \mathrm{n} \\
[p] = \dfrac{p!}{n! \cdot \pi} \cdot \binom{N}{p} \cdot \alpha^p \cdot [0] & \mathrm{n} \le \mathrm{p} \le \mathrm{N} \\
\pi = \prod_{v=1}^{p-n} (n + v \cdot \Theta) & \\
\sum_{p=0}^{N} [p] = 1 & \\
& \text{DELAY SYSTEM DISTRIBUTION} \\
& \text{OF ENGSET TYPE} \\
\alpha = \beta \cdot s & (\mathrm{N} > \mathrm{n}) \\
& (\Theta \ge 0)
\end{array}} \qquad \text{(TFD 3.2)}$$

For $\Theta = 0$ (TFD 3.2) is reduced to

$$\boxed{\begin{array}{ll}
[\,p\,] = \binom{N}{p} \cdot \alpha^p \cdot [\,0\,] & 0 \le p \le n \\
[\,p\,] = \dfrac{p!}{n! \cdot n^{p-n}} \cdot \binom{N}{p} \cdot \alpha^p \cdot [\,0\,] & n < p \le N \\
\sum_{p=0}^{N} [\,p\,] = 1 & \\
\alpha = \beta \cdot s & \\
\Theta = 0 &
\end{array}} \qquad \text{(TFD 3.3)}$$

<u>Time congestion</u>

For (TFD 3.2) and (TFD 3.3) we get

$$E = \sum_{p=n}^{N} [\,p\,] \qquad \text{(TFD 3.4)}$$

<u>Call Congestion</u>

For (TFD 3.2) and (TFD 3.3) the call congestion is equal to the probability that a call has to wait:

$$B = (\,P > 0\,) = \frac{\sum_{p=n}^{N} (\,N - p\,) \cdot \beta \cdot [\,p\,]}{\sum_{p=0}^{N} (\,N - p\,) \cdot \beta \cdot [\,p\,]}$$

$$B=\frac{\sum_{p=n}^{N}(N-p)\cdot[p]}{\sum_{p=0}^{N}(N-p)\cdot[p]} \qquad \text{(TFD 3.5)}$$

which can be written

$$B=\frac{(N-n)\cdot\Theta\cdot E+n\cdot(E-[n])}{\frac{\Theta+\beta}{1+\beta}\cdot N\cdot(1-E)+(N-n)\cdot\Theta\cdot E+n\cdot(E-[n])} \qquad \text{(TFD 3.5a)}$$

For $\Theta = 0$, B is reduced to

$$B=\frac{n\cdot(E-[n])}{\frac{N\cdot\alpha\cdot(1-E)}{1+\alpha}+n\cdot(E-[n])} \qquad \text{(TFD 3.6)}$$

For $\Theta = 1$

$$B=\frac{N\cdot E+n\cdot[n]}{N-n\cdot[n]}=\frac{E-\frac{n}{N}\cdot[n]}{1-\frac{n}{N}\cdot[n]} \qquad \text{(TFD 3.7)}$$

Note that the case $\Theta = 1$ gives

$$\left.\begin{aligned}[p]&=\binom{N}{p}\cdot b^{p}\cdot(1-b)^{N-p}\\ b&=\frac{\alpha}{1+\alpha}\\ \Theta&=1\end{aligned}\right\} \qquad \text{(TFD 3.8)}$$

If $p = n$, we obtain

$$[n]=\binom{N}{n}\cdot b^{n}\cdot(1-b)^{N-n}$$

(TFD 3.8) is identical to the Bernoulli distribution (TFL 2.1B) for a loss system. The state *[p]*, however, here signifies n occupations and $p - n$ waiting calls for $p > n$.

The expected proportion of calls that give up waiting is calculated as

$$B_{gu}=\frac{\sum_{p=n+1}^{N}[p]\cdot\frac{\Theta}{s}\cdot(p-n)}{\sum_{p=0}^{N}[p]\cdot(N-p)\cdot\beta}=\frac{\Theta}{\alpha}\cdot\frac{\sum_{p=n+1}^{N}[p]\cdot(p-n)}{\sum_{p=0}^{N}[p]\cdot(N-p)} \qquad \text{(TFD 3.9)}$$

which can be written

$$B_{gu}=\frac{\Theta}{\frac{\alpha}{1+\alpha}}\cdot\frac{(N\alpha-n\cdot(1+\alpha))\cdot E+n\cdot[n]}{N\cdot(\Theta+\alpha)+(N\alpha-n\cdot(1+\alpha))\cdot(\Theta-1)\cdot E+(\Theta-1)\cdot n\cdot[n]} \qquad \text{(TFD 3.9a)}$$

For $\Theta = 0$, $B_{gu} = 0$, which must be correct as $\Theta = 0$ implies that no call gives up.

For the case when $\Theta = 1$:

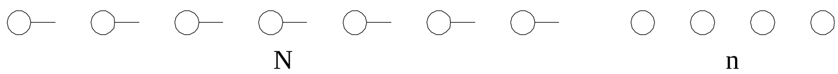
$$B_{gu} = E \cdot \left(1 - \frac{n}{N} \cdot \frac{1+\alpha}{\alpha} \right) + \frac{n \cdot [n]}{N \cdot \alpha} \qquad \text{(TFD 3.9b)}$$

The effect of Θ can be illustrated by the following numerical example:

Consider a group with $N = 10$ sources, $n = 4$ devices and $\alpha = 0.5$. For different values of Θ we obtain the following numerical values on the characteristics:

Characteristic	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = \infty$
E	0.511	0.441	0.408	0.289
B	0.397	0.350	0.326	0.240
B_{gu}	0	0.094	0.131	0.240
Q	0.571	0.313	0.223	0
A^1	3.143	3.020	2.962	2.755
A	3.143	3.333	3.408	3.623
$\frac{A-A^1}{A}$	0	0.094	0.131	0.240
$\frac{B_{gu}}{B}$	0	0.269	0.400	1.000

4. Infinite number of sources. Delay system of Erlang type (Erlang II)



N = ∞ n finite

$0 \leq p \leq \infty$

Assumption (TFD 1.1) + (TFD 1.6E) gives:

$$\left.\begin{array}{ll}
0 \leq p \leq \infty & \\
N = \infty & \\
n \text{ finite} & \\
\mu_p = \dfrac{p}{s} & 0 \leq p \leq n \\
\mu_p = \dfrac{n}{s} + \dfrac{\Theta}{s} \cdot (p-n) & n \leq p \leq \infty \\
\lambda_p = y(p) \cdot W(p) & \\
y(p) = y & \\
W(p) = 1 \quad \text{for all } p &
\end{array}\right\} \qquad \text{(TFD 4.1)}$$

$\Theta = 0$ means that all calls wait until served

$\Theta > 0$ means that waiting calls give up with the intensity $\frac{\Theta}{s}$

Using $\mu_p \cdot [p] = \lambda_{p-1} \cdot [p-1]$ (TGD 1.11) and applying the assumption (TFD 4.1), we get

$$\boxed{\begin{array}{ll} [p] = \dfrac{A^p}{p!}\cdot[0] & 0 \le p \le n \\ [p] = \dfrac{A^{p-n}}{\prod_{v=1}^{p-n}(n+v\cdot\Theta)}\cdot\dfrac{A^n}{n!}\cdot[0] & n < p \le \infty \\ \sum_{p=0}^{\infty}[p] = 1 & \\ A = y\cdot s \qquad \Theta \ge 0 & \end{array}} \tag{TFD 4.2}$$

For $\Theta = 0$

$$\boxed{\begin{array}{ll} [p] = \dfrac{A^p}{p!}\cdot[0] & 0 \le p \le n \\ [p] = \left(\dfrac{A}{n}\right)^{p-n}\cdot\dfrac{A^n}{n!}\cdot[0] & n \le p \le \infty \\ \sum_{p=0}^{\infty}[p] = 1 & \\ A = y\cdot s < n \qquad \Theta = 0 & \\ \textit{Erlang's distribution for DELAY system} & \end{array}} \tag{TFD 4.3}$$

Note - (TFD 4.3) requires that $A < n$ to satisfy the convergence condition (TGD 3.18).

Time congestion

$$E = \sum_{p=n}^{\infty}[p]$$

$$\boxed{\begin{array}{l} \textit{For } \Theta = 0 \\ E = \dfrac{\dfrac{A^n}{n!}\cdot\dfrac{n}{n-A}}{\sum_{v=0}^{n-1}\dfrac{A^v}{v!}+\dfrac{A^n}{n!}\cdot\dfrac{n}{n-A}} = D_n(A) \\ \textit{which is Erlang's Second Formula} \end{array}} \tag{TFD 4.4}$$

For $\Theta > 0$ the expression is more complicated but fully calculable.

For the special case $\Theta = 1$, we obtain

$$[p] = \frac{A^p}{p!}\cdot e^{-A} \qquad 0 \le p \le \infty$$

and

$$E = \sum_{p=n}^{\infty}\frac{A^p}{p!}\cdot e^{-A} \tag{TFD 4.5}$$

The case $\Theta = 1$ is identical to the Poisson distribution for a full availability group in a loss system (compare section TFL). The state (p), however, here signifies n occupations and $p - n$ waiting calls when $p > n$.

Call congestion

According to (TFD 2.2),

$$B = P(>0) = \frac{\sum_{p=n}^{\infty} y[p]}{\sum_{p=0}^{\infty} y[p]} = \sum_{p=n}^{\infty} [p] = E$$

i.e. $\boxed{B = E}$ (TFD 4.6)

as $y\,(p) = y$ is independent of the state of the system, (p). (Infinite number of sources).

For $\Theta > 0$ the proportion of calls which give up waiting is calculated as

$$B_{gu} = \frac{\sum_{p=n+1}^{\infty} [p] \cdot \frac{\Theta}{s} \cdot (p-n)}{\sum_{p=0}^{\infty} [p] \cdot y} = \frac{\Theta}{A} \cdot \sum_{p=n+1}^{\infty} [p] \cdot (p-n) \qquad \text{(TFD 4.7)}$$

$$A = y \cdot s$$

From

$$(n + \Theta(p-n)) \cdot [p] = A \cdot [p-1]$$

or

$$(p-n) \cdot [p] = \frac{A}{\Theta} \cdot [p-1] - \frac{n}{\Theta} \cdot [p]$$

and

$$\sum_{p=n+1}^{\infty} (p-n) \cdot [p] = \frac{A}{\Theta} \cdot \sum_{p=n}^{\infty} [p] - \frac{n}{\Theta} \cdot \sum_{p=n+1}^{\infty} [p] \qquad \text{(TFD 4.8)}$$

we get

$$\sum_{p=n+1}^{\infty} (p-n) \cdot [p] = \frac{1}{\Theta} \cdot \left(A \cdot E - n \cdot E + n \cdot [n] \right)$$

so that (TFD 4.7) can be written

$$B_{gu} = \frac{n}{A} \cdot [n] + \frac{A-n}{A} \cdot E \qquad \text{(TFD 4.7a)}$$

where $[n]$ and E are dependent on Θ.

For $\Theta = 1$

$$B_{gu} = \frac{A^n}{n!} \cdot e^{-A} - \frac{n-A}{A} \cdot \sum_{p=n+1}^{\infty} \frac{A^p}{p!} \cdot e^{-A} \qquad \text{(TFD 4.9)}$$

In (TFD 4.7a) and (TFD 4.8) A can be $< n$ or $> n$.

Traffic carried and traffic offered

According to (TFD 2.3), the traffic carried is

$$A^1 = \sum_{p=1}^{n-1} p \cdot [p] + \sum_{p=n}^{\infty} n \cdot [p] = A \cdot \sum_{p=1}^{n-1} [p-1] + n \cdot \sum_{p=n}^{\infty} [p]$$

$$A^1 = A \cdot \left(1 - \sum_{p=n+1}^{\infty} [p] \right) + n \cdot E \qquad (\mathrm{A} = \mathrm{y} \cdot \mathrm{s})$$

$$A^1 = A \cdot \left(1 - [n-1] - E \right) + n \cdot E$$

$$A^1 = A + (n - A) \cdot E - n \cdot [n] \qquad \text{(TFD 4.10)}$$

We also note that

$$A \cdot (1 - B_{gu}) = A + (n - A) \cdot E - n \cdot [n]$$

For the offered traffic according to (TFD 2.4)

$$A = \sum_{p=0}^{\infty} s \cdot y[\,p\,] = s \cdot y \qquad \text{(TFD 4.11)}$$

The difference between A and A^1 is

$$\Delta A = A - A^1 = n \cdot [n] - (n - A) \cdot E \qquad \text{(TFD 4.12)}$$

For $\Theta = 0$

$$\boxed{\begin{array}{l} E = D_n(A) = \dfrac{\dfrac{A^n}{n!} \cdot \dfrac{n}{n-A}}{\sum\limits_{v=0}^{n-1} \dfrac{A}{v!} + \dfrac{A^n}{n!} \cdot \dfrac{n}{n-A}} = \dfrac{A^n}{n!} \cdot \dfrac{n}{n-A} \cdot [0] \\ [n] = \dfrac{A^n}{n!} \cdot [0] \qquad\qquad D_n(A) = \text{Erlang's Second Formula!} \end{array}} \qquad \text{(TFD 4.4a)}$$

Consequently

$$\frac{n}{n-A} \cdot [n] = E$$

or

$$n \cdot [n] = (n - A) \cdot E$$

from which it follows that

$$\begin{array}{l} \Delta A = 0 \\ A = A^1 \\ \Theta = 0 \end{array} \qquad \text{(TFD 4.13)}$$

For $\Theta > 0$, on the other hand,

$$A > A^1 \qquad \text{(TFD 4.14)}$$

According to (TFD 2.6) and (TFD 4.9) the mean length of queues = the mean number of waiting calls in the systems.

$$\left.\begin{aligned} &Q = \sum_{p=n+1}^{\infty}(p-n)\cdot[p] = \frac{1}{\Theta}\cdot\big(n\cdot[n]-(n-A)\cdot E\big) \\ &\Theta > 0 \end{aligned}\right\} \qquad \text{(TFD 4.15)}$$

For $\Theta = 0$

$$\left.\begin{aligned} &Q = \frac{A}{n-A}\cdot D_n(A) \\ &\Theta = 0 \\ &A < n \end{aligned}\right\} \qquad \text{(TFD 4.16)}$$

<u>Probability that x specified devices are engaged</u>

For random hunting and Θ = 0, we get, according to (TDG 3.11)
(after some rearrangement)

$$\left.\begin{aligned} &H(x) = \frac{n-A}{n}\cdot D_n(A)\cdot\left(\frac{1}{E_{n-x}(A)}+\frac{A}{n-A}\right) \\ &\Theta = 0 \end{aligned}\right\} \qquad \text{(TFD 4.17)}$$

where $E_n(A)$ is Erlang’s first formula (TFL 3.1E). The relation between $D_n(A)$ and $E_n(A)$ can be written:

$$\begin{aligned} &D_n(A) = \frac{n\cdot E_n(A)}{n-A\cdot\big(1-E_n(A)\big)} \\ &(A < n) \end{aligned} \qquad \text{(TFD 4.18)}$$

or

$$\left.\begin{aligned} &E_n(A) = \frac{(n-A)\cdot D_n(A)}{n-A\cdot D_n(A)} \\ &A < n \qquad \Theta = 0 \end{aligned}\right\} \qquad \text{(TFD 4.18a)}$$

<u>Load on the ν:th device</u>

As $N = \infty$, the load on the <u>ν</u>:th device in a <u>sequentially hunted</u> full availability group can be calculated as

$$\begin{aligned} &a_\nu = b_\nu + (1-b_\nu)\cdot\frac{A}{n}\cdot D_n(A) \\ &b_\nu = A\cdot\big(E_{\nu-1}(A)-E_\nu(A)\big) \\ &\Theta = 0 \end{aligned} \qquad \text{(TFD 4.19)}$$

<u>For random hunting</u>

$$\left.\begin{aligned} &a_\nu = \frac{A'}{n} = \frac{A}{n}+\frac{n-A}{n}\cdot E-[n] \\ &\Theta > 0 \end{aligned}\right\} \qquad \text{(TFD 4.20)}$$

and

$$\left.\begin{aligned} &a_\nu = \frac{A}{n} \\ &\Theta = 0 \end{aligned}\right\} \qquad \text{(TFD 4.20a)}$$

The effect of different Θ - values can be illustrated by the following numerical example for N = 4, A = 3:

Characteristic	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = \infty$
$E = B$	0.509	0.353	0.314	0.206
B_{gu}	0	0.107	0.133	0.206
Q	1.528	0.319	0.199	0
A^1	3	2.681	2.601	2.382
A	3	3	3	3
$\frac{A\text{-}A^1}{A}$	0	0.107	0.133	0.206
$\frac{B_{gu}}{B}$	0	0.302	0.424	1.000

5. Waiting time distribution

The mean waiting (U) is usually independent of the queue discipline, whereas the waiting time distribution $f(t)$ is strongly depending on it. There is no general method for derivation of U and $f(t)$ and the expressions for them are not usually simple explicit expressions.

The simplest cases are obtained with ordered queuing, the very simplest being that of Erlang' s waiting time distribution (TFD 4.3). We shall limit ourselves to relating this case.

For (TFD 4.3) the mean waiting time for waiting calls is

$$u = \frac{s}{n - A} \qquad \text{(TFD 5.1)}$$

and for all calls

$$U = P(>0) \cdot u = D_n(A) \cdot \frac{s}{n - A} \qquad \text{(TFD 5.2)}$$
$$\Theta = 0$$

The waiting time distribution for waiting calls will be according to (TFD 2.11)

$$P_v(>t_0) = e^{-\frac{n-A}{s} \cdot t_0} \qquad \text{(TFD 5.3)}$$

and for all calls

$$\left.\begin{aligned} P(\tau = 0) &= 1 - D_n(A) \\ P(\tau > t_0) &= D_n(A) \cdot e^{-\frac{n-A}{s} \cdot t_0} \end{aligned}\right\} \qquad \text{(TFD 5.4)}$$

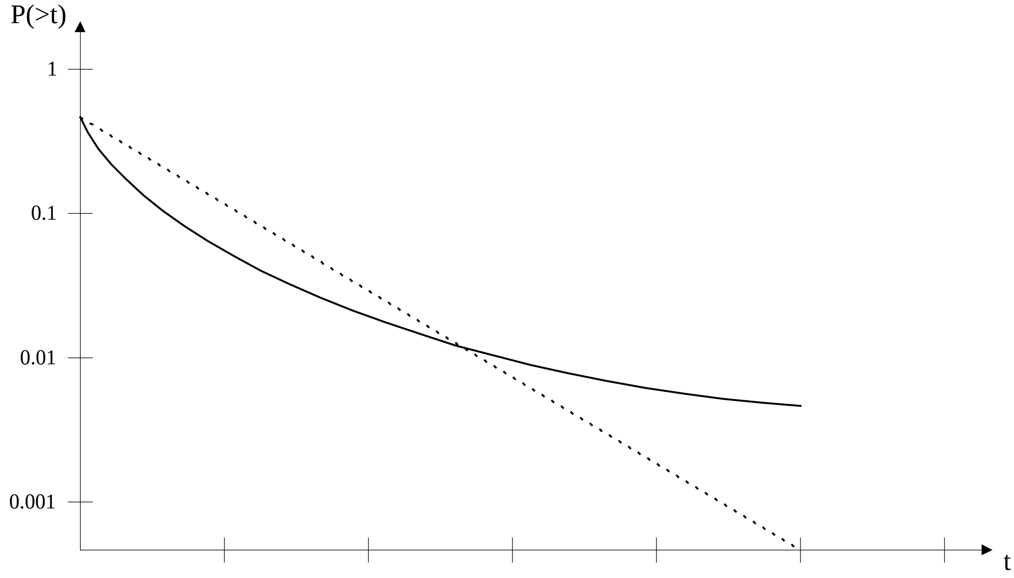


FIGURE TFD 5/1

Waiting time distribution for all calls,for ordered queue (dotted) and for randomly serviced queue.
The two cases have the same mean value.
No waiting calls give up waiting ($\Theta = 0$).

6. <u>Delay system with constant holding time: the Crommelin distribution</u>

For cases with constant holding time the assumptions made in section TGD do not apply. To derive cases of this kind, one can use Fry' s equations of state.

Fry's equations of state. Concept.

Assume equilibrium and consider the system at the points of time t and $t + h$, where h is the constant holding time

The occupations which were in progress at t are all completed at $t + h$, at $t + h$ there are now only calls which were queuing at t and new calls.

$$[p]_{t+h} = \sum_{v=0}^{n} [v]_t \cdot P(p,h) + \sum_{v=n+1}^{n+p} [v]_t \cdot P(p - v + n,h) \qquad \text{(TFD 6.1)}$$

$P(p,h)$ = probability that exactly p calls occur during the interval $(t, t + h)$.

For $N = \infty$, $\Theta = 0$ and ordered queue, Crommelin' s distribution is obtained.

For n = 1 the mean waiting time for all calls is

$$U = \frac{h}{2} \cdot \frac{a}{1 - a} \qquad \text{(TFD 6.2)}$$

a = traffic offered = traffic carried < 1.

For $n > 1$ the exact solution will be very complicated. One can, however, use a modification of Molina's approximate expression:

$$U \cong \frac{h}{n+1} \cdot \frac{D_n(A)}{1-a} \cdot \frac{1-a^{n+1}}{1-a^n} \cdot \frac{1}{1+a} \qquad \text{(TFD 6.3)}$$

$$a = \frac{A}{n} < 1 \qquad n \geq 1$$

Note that (TFD 6.2), for constant holding time, gives exactly half of the mean waiting time obtained for exponentially distributed holding time with the same mean holding time. The expression (TFD 5.2) gives for

$$n = 1 \quad A = a \quad s = h$$

$$U = h \cdot \frac{a}{1-a} \qquad \text{(TFD 6.4)}$$

(exponential distribution)

For $n = 1$

$$U_{const} = \frac{1}{2} \cdot U_{exp} \qquad \text{(TFD 6.5)}$$

and for arbitrary n according to (TFD 6.3) and (TFD 5.2):

$$U_{const} = U_{exp} \cdot \frac{n}{n+1} \cdot \frac{1-a^{n+1}}{1-a^n} \cdot \frac{1}{1+a} \qquad \text{(TFD 6.6)}$$

i.e.

$$U_{const} < U_{exp}$$

The waiting time distribution for the Crommelin case can be written:

$$\left.\begin{aligned} P(t=0) &= 1 - D_n(A) \\ P(t>t_0) &= \sum_{\mu=1}^{\infty} \sum_{v=0}^{n-1} P_v \cdot \frac{(A \cdot B)^x}{x!} \cdot e^{-A \cdot B} \end{aligned}\right\} \qquad \text{(TFD 6.7)}$$

where

$$\left\{\begin{aligned} B &= \mu - \left(\frac{t_0}{h} - \left\{ \frac{t_0}{h} \right\} \right) \\ x &= \left(\mu + \left\{ \frac{t_0}{h} \right\} \right) \cdot n + n - 1 - v \\ P_v &= \sum_{p=0}^{v} [\,p\,] \\ \left\{ \frac{t_0}{h} \right\} &= \textit{the integer part of } \frac{t_0}{h} \\ A &= \textit{traffic offered} < n \end{aligned}\right.$$

The probability $P(t>t_0)$ of long waiting times will be less for (TFD 6.7) than for (TFD 5.4).

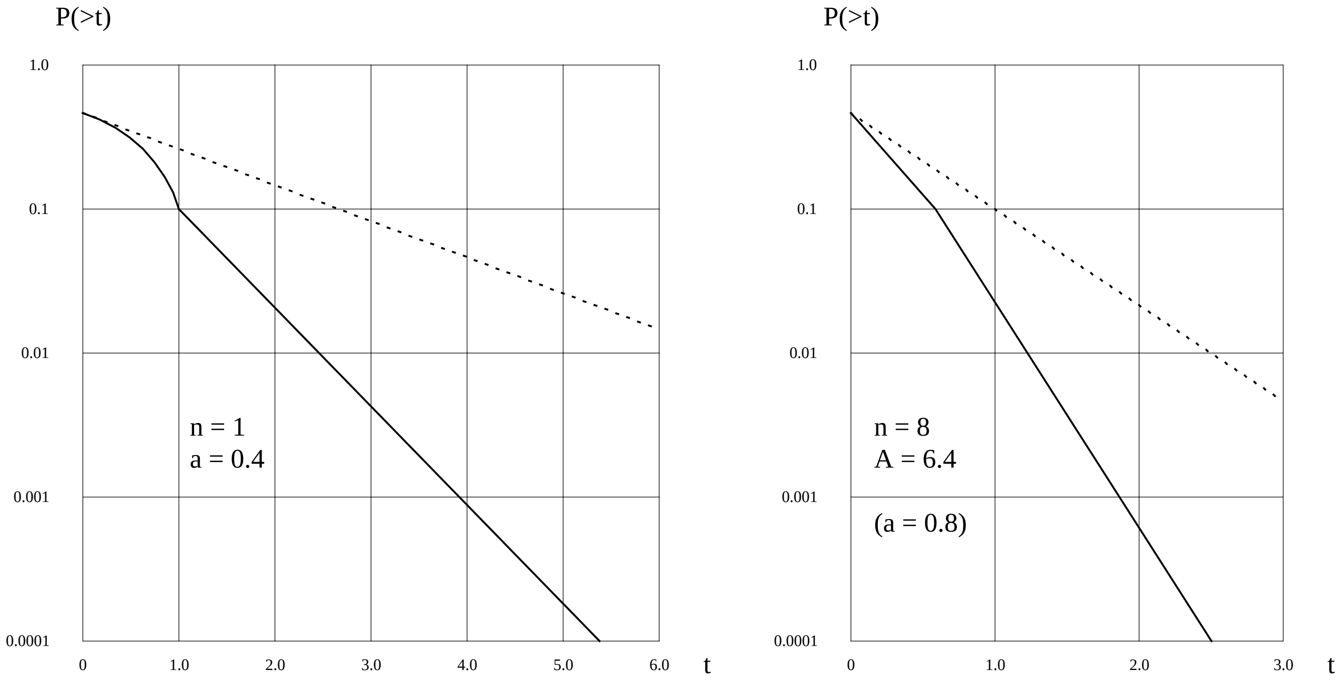


FIGURE TFD 6/1 : Waiting time distribution for all calls for exponentially distributed holding time (dotted) and for constant holding time.
Infinite number of sources and ordered queue; no waiting calls give up *(θ = 0)*.
The time *t* is expressed in multiples of the constant holding time.

7. Composite holding time distribution

In many practical cases the holding time distributions are neither constant nor exponentially distributed. Often there is a composition of different constant holding times. Often, too, some of these holding times are prolonged by the necessity to wait for other devices. This waiting time may often have the character of an approximate distribution.

If one determines the real holding time distribution with its mean h and variance σ^2, one may usually expect that, in practical cases, the values of σ/h will be between the corresponding values for the cases with constant and exponential waiting time for all calls ($\theta = 0$).

<u>Infinite number of sources ($N = \infty$)</u>

$$U \approx \left(1 - \frac{\sigma^2}{\bar{h}^2}\right) \cdot U_{const} + \frac{\sigma^2}{\bar{h}^2} \cdot U_{\exp} \qquad \text{(TFD 7.1)}$$

where

$\bar{h} = \int_{t=0}^{\infty} t \cdot f(t)\, dt$ is the mean holding time

$\sigma^2 = \int_{t=0}^{\infty} (t - \bar{h})^2 \cdot f(t)\, dt$ is the variance

U_{const} from (TFD 6.3) (modified Molina)

$U_{\exp}$ from (TFD 5.2) (Erlang)

$\bar{h}$ and σ^2 from the holding time distribution

For constant holding time we have $\sigma^2 = 0$ and $U = U_{const}$.

For an exponential holding time distribution we have $\sigma^2 = \bar{h}^2$ and $U = U_{\exp}$.

U is unaffected by the queue discipline.

<u>Finite number of sources</u>

With a finite number of sources, the waiting times are limited as compared with $N = \infty$. An approximate method for taking this effect into account, with composite holding time distribution and finite number of sources, is to use the following formula for the mean waiting time for all calls:

$$U(n, N) = \left\{\left(1 - \frac{\sigma^2}{\bar{h}^2}\right) \cdot U_{const} + \frac{\sigma^2}{\bar{h}^2} \cdot U_{\exp}\right\} \cdot \left(1 - \frac{n}{N}\right) \qquad \text{(TFD 7.2)}$$

where

$$\begin{cases} U_{const} & \text{from (TFD 6.3) (modified Molina)} \\ U_{\exp} & \text{from (TFD 5.2) (Erlang)} \\ \bar{h} \text{ and } \sigma^2 & \text{from the holding time distribution} \end{cases}$$

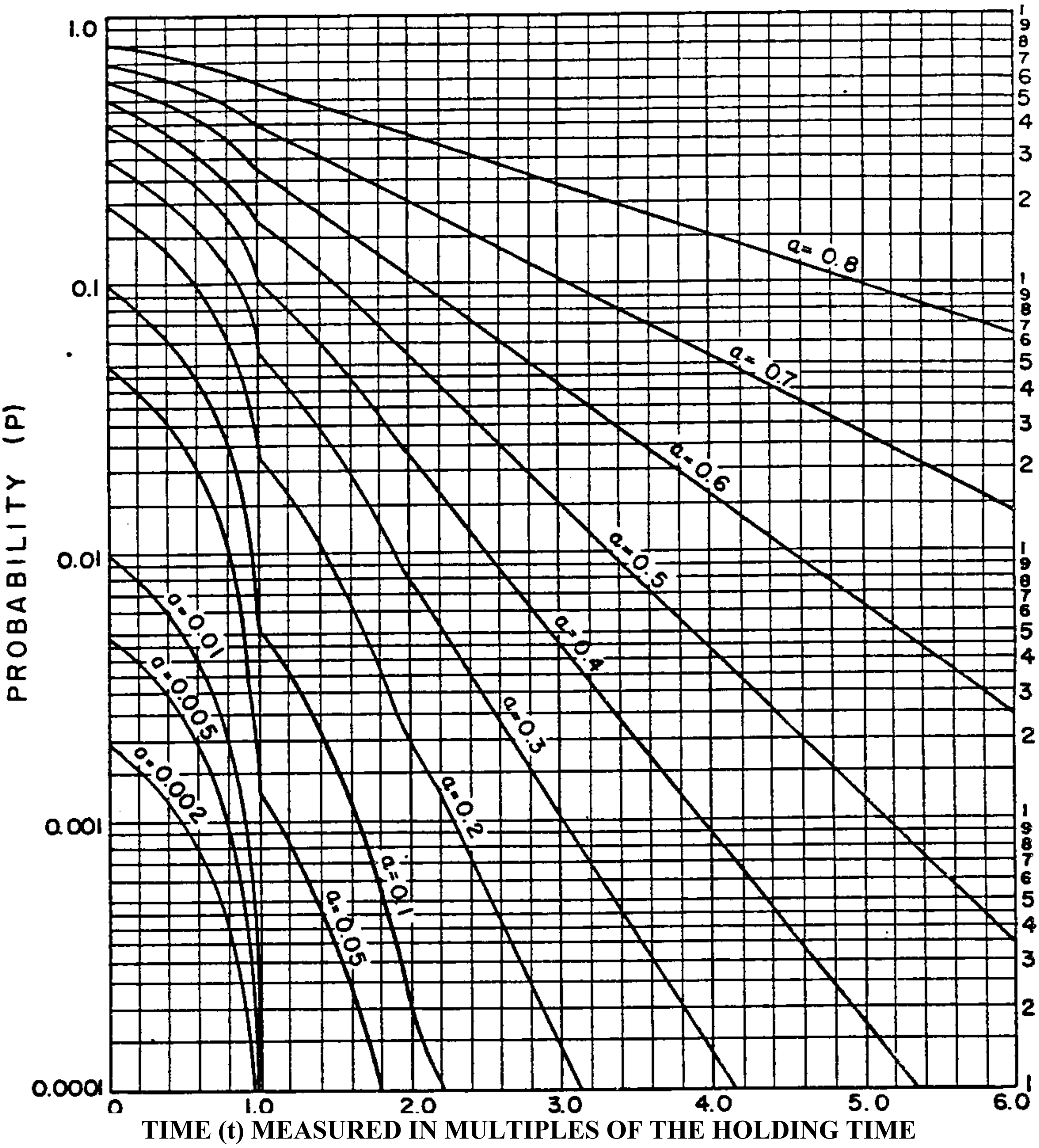


FIGURE A

Probability *P* for a delay exceeding *t*

when there are *n = 1* switches with occupancy α

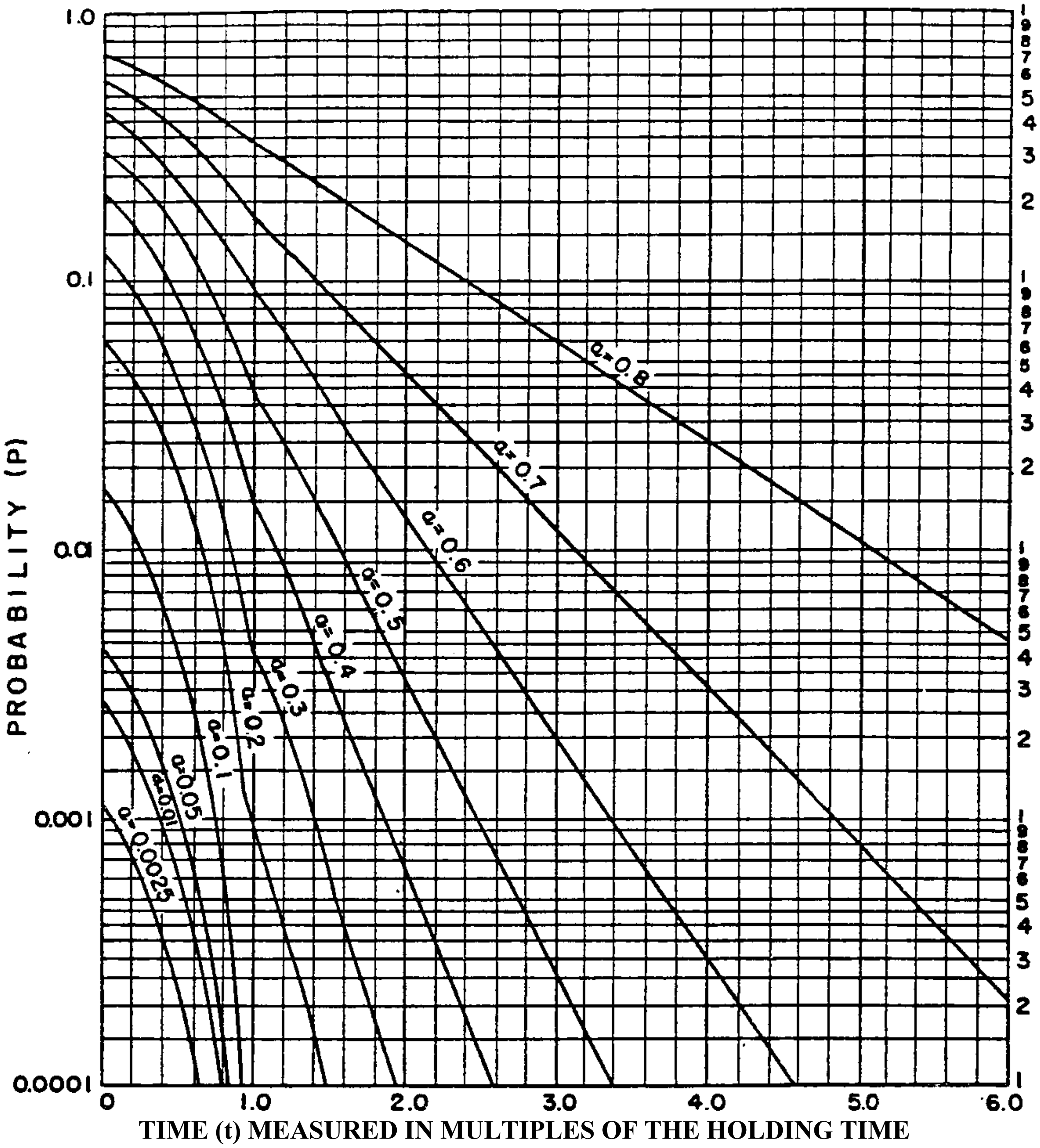


FIGURE B

Probability P for a delay exceeding t

when there are $n = 2$ switches with occupancy α

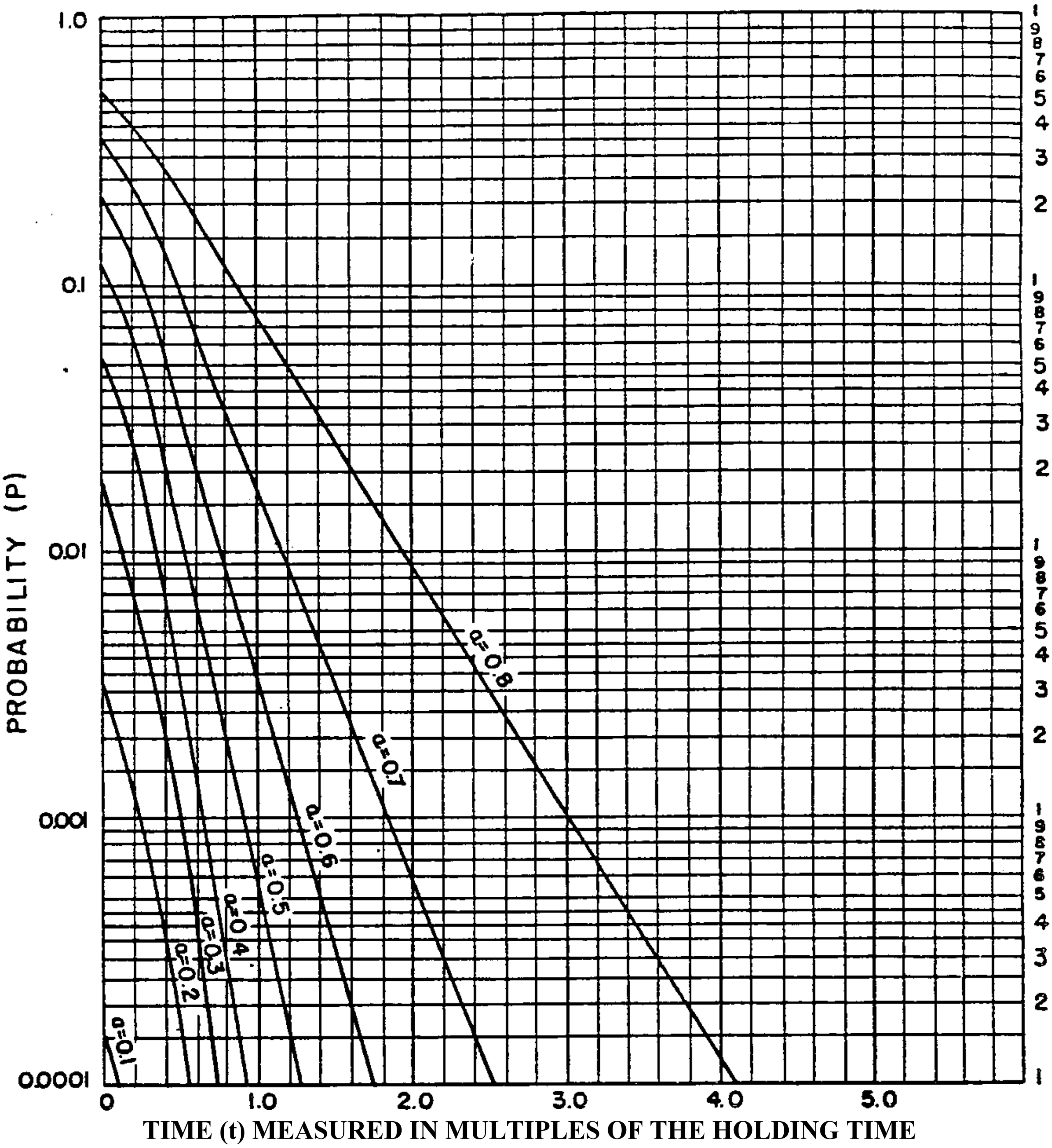


FIGURE C

Probability *P* for a delay exceeding *t*

when there are *n = 5* switches with occupancy α

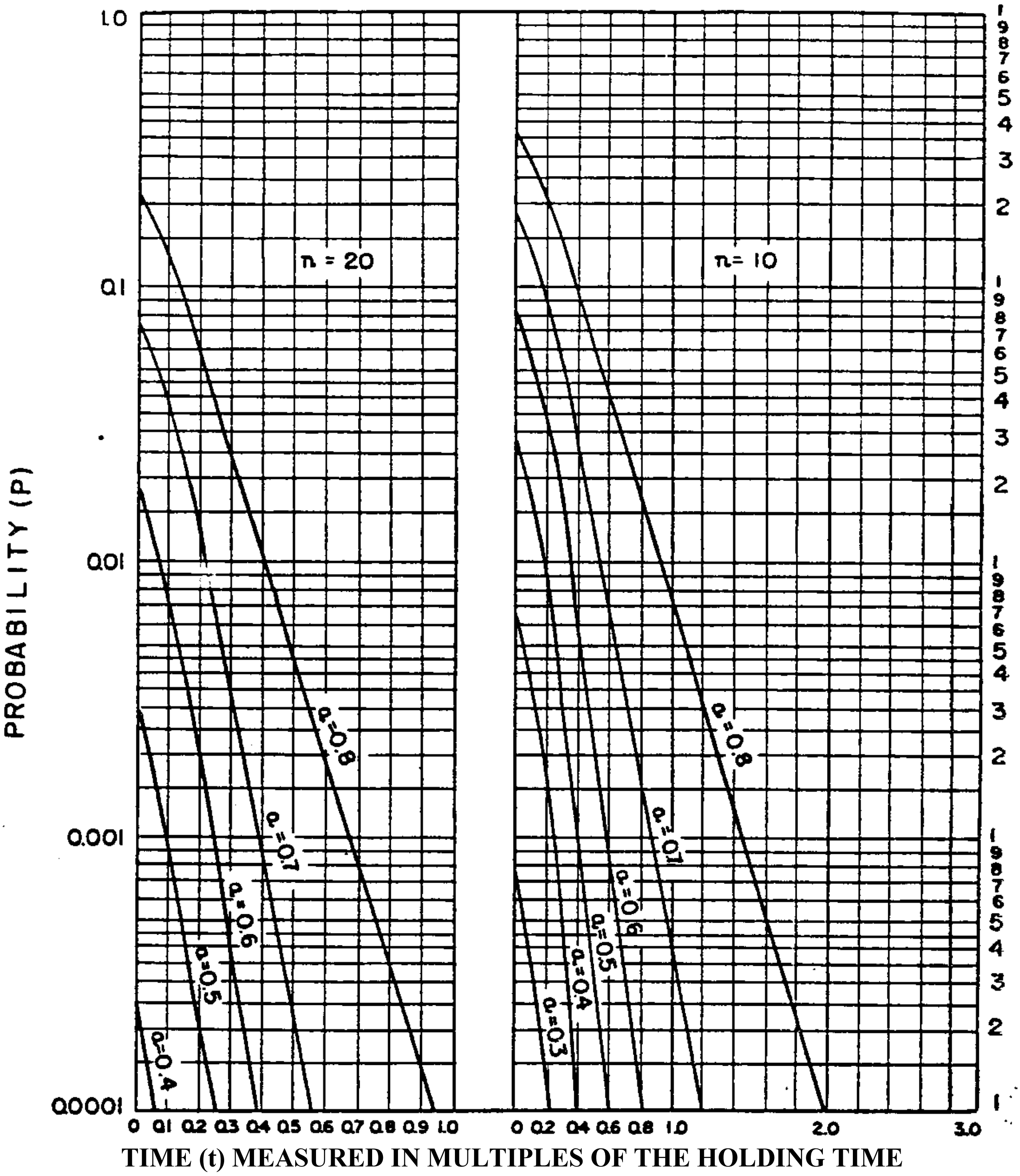


FIGURE D

Probability *P* for a delay exceeding *t*

when there are *n = 10* and *n = 20* switches with occupancy α

FIGURE TFD 7/1

CROMMELIN DISTRIBUTION. GRAPHS SHOWING THE PROBABILITY OF HAVING TO WAIT LONGER THAN A GIVEN TIME. THIS TIME BEING EXPRESSED IN MULTIPLES OF THE CONSTANT HOLDING TIME

Waiting time distribution

Expressions for the waiting time distribution for a composite holding time distribution are usually very complicated. The queue discipline, furthermore, has a great significance. The probability of very short and very long waiting times is greater with a random than with an ordered queue.

With a priority queue the result is roughly the same as in cases with a finite number of sources where the number of sources = the priority.

8. General comments; Kendall's notations

During the last decades an enormous amount of theoretical studies has been made on delay systems. These studies, which use different combinations of assumptions regarding input, service time and queue discipline, are mainly dealing with the description of waiting cases outside the telecommunications field. It is estimated that there may be at least 1000 articles and 100 books on queuing problems available through libraries today. It is quite clear that this wave of interest for queuing problems has been beneficial and fruitful for the treatment of waiting problems within the telecommunications field as well. However, most of the solutions presented involve rather complicated mathematical expressions which have to be used for arriving at practical results. Therefore, the long known formulas deduced by Erlang, Molina, Fry, Crommelin and others are still of the highest value when it comes to practical calculations.

To bridge the gaps where existing simple formulas cannot provide accurate practical results, the traffic engineer of today generally prefers to use computer simulations. Provided the real conditions are correctly reflected in the assumptions transferred to the design and dimensioning of waiting systems used in telecommunication plant. The simulations also advise when simple approximate formulas can be applied and when they should not be used.

In order to easily define the assumptions used in a delay system study, D.G. Kendall introduced a notation system which by code letters describes the assumptions for

a) the input process
b) the holding/service times
c) the number of servers
d) the queue discipline.

Since notations, like M/M1 (FIFO) are used frequently in the literature, it is reason to relate them here.

Therefore, a classification of a queuing system

a/ b/ c/ (d)

For the input process, *a*, the following notations are used:

M =Exponentially distributed intervals between calls, i.e. Poisson input, (infinite number of sources)

D = Deterministic input, i.e. constant intervals between calls;

G = General distribution for intervals between calls;

GI = General independent intervals;

E_k = Erlang-k distributed intervals (where $E_1 \equiv M$).

For the holding time distribution, b), the same letters as for a) are used, then defining the holding time distribution.

For the <u>number of servers</u>, c), the notation is usually

1 for one server,

n for an arbitrary number of servers.

For the <u>queue discipline</u>, d) example of typical notations are:

FIFO: First in, First out (ordered queue)

RANDOM: Random queue.

Consequently, as an example of Kendall's classification system, M/ M/n (FIFO0 defines the Erlang distribution for delay systems (defined in (TFD 4.3), (TFD 5.1) - (TFD 5.4)) as a full availability group with an arbitrary number of devices (n), and when no callers give up waiting. In the same way, M/D/n (FIFO) defines the Crommelin waiting time distribution for Poisson input, constant holding times and ordered queue as described in Chapter TFD 3.

The classification system does not give a complete picture of the considered system since the following details are missing from the Teletraffic Theory point of view:

- number of single traffic sources generating the traffic input;
- grouping arrangements other than full availability, as for instance: gradings, link systems, etc.;
- hunting rules whenever the hunting for free devices has an impact on the traffic case;
- delayed calls waiting until served, or not.

This section has only dealt with full availability groups where there is only one possible waiting stage, namely outside the group before the calls being served. The telecommunication techniques provide, however, also more complicated waiting cases, especially when it comes to processing calls in markers, processors and other common equipment. In a setting-up procedure for a call, waiting may namely occur in different stages of this procedure, meaning that the total setting-up time may consist of the sum of working times plus a number of possible waiting times. This type of waiting system is called Composite Delay System.

Basic Teletraffic Theory (T)

EXERCISES B

Full Availability Group, Delay System

Basic Exercises

TXB 1

To a full availability group in a delay system consisting of 30 devices is offered a traffic with the calling rate 700 calls/hour and the mean holding time 108 sec.

Supposing the assumptions for the Erlang distribution for delay systems to be fulfilled, calculate

a) the traffic offered;
b) the probability of delay;
c) the traffic handled;
d) the mean of the traffic handled per device;
e) the mean waiting time for all calls;
f) the mean waiting time for calls, which have to wait.

TXB 2

Same assumptions and numerical values as in Exercise TXB 1. Assume further that the queue is served in the order of arrival.

a) Calculate the probability that a call has to wait longer than 3, 6, 12, 24, sec.;
b) Calculate the probability that a call has to wait longer than 3, 6, 12, 24 sec., given that it has to wait.

TXB 3

It is assumed that calls arrive to 1 device in a delay system according to a Poisson process with calling rate 4,320 calls/hour.

The holding times are assumed to be independent of one another and of the arrival process. They are assumed to follow an exponential distribution with the mean holding time S sec.

It is further assumed that no queuing calls give up waiting and that queuing calls are served in the order of arrival.

What is the largest value of S in sec. that can be allowed to fulfilled the condition that the probability that a call has to wait longer than 3 sec. shall not be greater than 0.01?

TXB 4

What is the greatest calling rate that can be offered to a full availability group of 2 devices in a delay system if the mean waiting time for all calls shall be at most 2 sec.?

The mean holding time is known to be 0.4 sec. and the assumptions of the Erlang distribution for delay systems are assumed to be fulfilled.

TXB 5

Show that

$$E_1\ (A) < E_2\ (A)$$

TXB 6

Consider an Erlang delay system with 10 trunks and offered traffic 8 erlangs. The mean conversation time is 5 min. Delayed calls are served in order of arrival. What is the probability that an arriving call finding one other call in the queue has to wait longer than 0.5 min.?

TXB 7

Consider an Erlang delay system with $n = 10$ trunks and $A = 4$ erl. Calculate the proportion of time when there is at least one call waiting.

TXB 8

To 1 device in a delay system calls are arriving according to a Poisson process with the calling rate 4,320 calls/hour.

The holding times are independent random variables with one and the same distribution:

$$\mathrm{P}\left\{\text{holding time } = \text{ 0.2 sec.}\right\} = \frac{1}{3}$$

$$\mathrm{P}\left\{\text{holding time } = \text{ 0.4 sec.}\right\} = \frac{2}{3}$$

The holding times and the inter-arrival times are independent and no queuing calls give up waiting.

a) What is the probability of delay?
b) What is the mean waiting time for calls which have to wait?
c) What is the mean waiting time for all calls?

TXB 9

1 device in a delay system is considered.

It is assumed that calls are arriving according to a Poisson process with the calling rate 2,880 calls/hour.

The holding times are assumed to be independent random variables with a distribution given by the frequency function

$$f(t) = \begin{cases} 0 & (t \leq 0) \\ 16 \cdot t \cdot e^{-4 \cdot t} & (t > 0) \end{cases}$$

where t is measured in sec.

The holding times and the inter-arrival times are assumed to be independent of one another and it is further assumed that no queuing calls give up waiting.

a) What is the probability of delay?
b) What is the mean waiting time for calls which have to wait?
c) What is the mean waiting time for all calls?