

General Description of Traffic

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Basic Teletraffic Theory (T)

GENERAL DESCRIPTION OF TRAFFIC (TGD)

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1. Common theory for different cases

For the mathematical description of traffic processes, use is made, in principle, of stochastic processes of the type usually called "birth and death processes". To arrive at practically usable results, one must generally assume that the birth and death intensities are independent of time, and deal with the process when it has reached equilibrium.

For purposes of telephone traffic engineering this process, the "traffic process" describes the number of busy devices or single sources as a function of time. This leads to general equations of state from which, in the limit, when the time $t \rightarrow \infty$, one can derive the state probabilities for the system. It is found that from these equilibrium equations for the state probabilities, one can set up abbreviated expressions which well cover the most general cases within traffic theory. It is there that we shall start.

We make the following basic assumptions:

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| <p>(1) Statistical equilibrium, which implies:
 INCREASE = DECREASE
 in the long run ($t \rightarrow \infty$)</p> <p>(2) Probability distribution for time intervals between successive calls.</p> <p>(3) Probability distribution for the duration of the individual occupations.</p> <p>(4) The individual traffic source acts independently of the state of the other sources.</p> <p>(5) The duration of the individual occupation is independent of other occupations.</p> <p>(6) Deterministic rules or probability distributions applied for what happens to unsuccessful calls.</p> |
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(TGD 1.1)

The theory will be especially simple if a negative exponential distribution of suitable type is assumed for 2) and 3).

The traffic is considered to be generated by single sources which can make only one call at a time. The single source has two possible states:

- 0. Free
- 1. Engaged

The number of single sources (called "sources" in the sequel) may be either finite or infinite, but the traffic they generate must be finite.

The use of the exponential distribution for time intervals, or durations, implies that one has the frequency function

$$f(t) = \gamma \cdot e^{-\gamma \cdot t} \quad (0 \leq t \leq \infty) \quad (\text{TGD 1.2})$$

and the distribution function

$$F(t) = 1 - e^{-\gamma \cdot t} = P(\leq t) \quad (\text{TGD 1.3})$$

The mean value is

$$\bar{t} = \int_{t=0}^{\infty} t \cdot f(t) \cdot dt = \frac{1}{\gamma} \quad (\text{TGD 1.4})$$

The inverse distribution function

$$\varphi(t) = 1 - F(t) = e^{-\gamma \cdot t} = P(> t) \quad (\text{TGD 1.5})$$

yields the probability that the time interval is greater than t.

The probability that it terminates in the interval (t, t + Δ t) is now:

$$\int_t^{t+\Delta t} f(\tau) d\tau = F(t + \Delta t) - F(t) = \varphi(t) - \varphi(t + \Delta t) = e^{-\gamma \cdot t} - e^{-\gamma(t+\Delta t)} \quad (\text{TGD 1.6})$$

The probability that the “duration” is at least t is:

$$\varphi(t) = e^{-\gamma \cdot t} \quad (\text{TGD 1.7})$$

The conditional probability that a “duration” which still exists at t ceases in (t, t + Δ t) is now

$$P(\Delta t) = \frac{e^{-\gamma \cdot t} - e^{-\gamma \cdot (t+\Delta t)}}{e^{-\gamma \cdot t}} = 1 - e^{-\gamma \cdot \Delta t} \quad (\text{TGD 1.8})$$

Expansion of (TGD 1.8) gives

$$P(\Delta t) = \gamma \cdot \Delta t - \frac{\gamma^2 \cdot \Delta t^2}{2!} + \frac{\gamma^3 \cdot \Delta t^3}{3!} - \dots \quad (\text{TGD 1.8a})$$

For small values of Δt, the first term dominates and one may say that the probability of an event in (t, t + Δt) is equal to:

$$P(\Delta t) \approx \gamma \cdot \Delta t \quad (\text{TGD 1.9})$$

i.e. proportional to the intensity of change, γ and to the time interval Δt.

The assumption of exponential distribution implies:

- 1) P (Δt) will be independent of t.
- 2) The process assumes that only one event can take place at a time, since the probability of two events would be proportional to (Δt)², which can be disregarded when Δt → 0.

The momentary state of a system is described by the number of simultaneously engaged sources. Usually, this number is synonymous to the number of engaged devices.

The momentary states of the system are denoted by (p). The probabilities of these states, the probabilities of state are denoted by [p], (p ≥ 0).	(TGD 1.10)
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[p] is also equal to the part of the time during which the system is in state (p).

On the assumption of statistical equilibrium, the system must change:

$$(p - 1) \rightarrow (p)$$

as often as $(p) \rightarrow (p-1)$

Otherwise, the system would not retain its equilibrium but in the long run would tend to either p = 0 or p = max.

Denote the intensity of increase by λ_p and the intensity of decrease by μ_p when the system is in (p) and assume exponential distribution of intervals between calls and of holding times.

Increase = Decrease

then gives:

$$\lambda_{p-1} \cdot [p-1] = \mu_p \cdot [p] \quad (\text{TGD 1.11})$$

for all possible $p > 0$.

The system must always be in one of these possible states (p). Therefore:

$$\sum_p [p] = 1 \quad (\text{TGD 1.12})$$

By recursion from $p = 1$ upwards, we get:

$$[p] = \frac{\prod_{\rho=0}^{p-1} \lambda_{\rho}}{\prod_{\rho=1}^p \mu_{\rho}} \cdot [0]$$

$$\sum_p [p] = 1 \quad (\text{TGD 1.13})$$

which is the general solution for a traffic process in statistical equilibrium with exponential distribution of changes.

The physical counterpart of (TGD 1.11) - (TGD 1.13) for the full availability group can be represented by a group with N sources and n devices (Figure TGD 1/1).

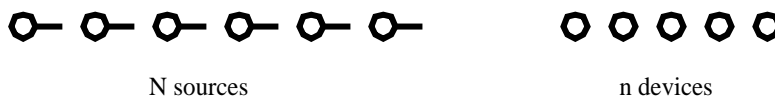


Figure TGD 1/1 : Full availability group

Here p engaged sources correspond to p engaged devices (if $p \leq N$, $p \leq n$).

For a grading and a link system, the representation will be somewhat more complicated (Figure TGD 1/2 and Figure TGD 1/3).

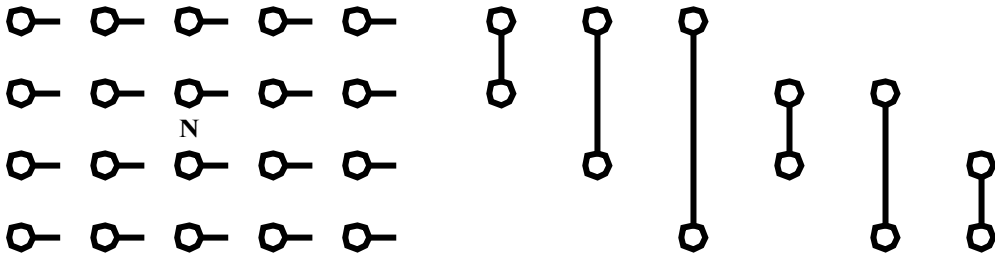


Figure TGD 1/2 - Grading

In Figure TGD 1/2, the $N = 20$ sources are divided into four inlet groups, each of which has access to 3 of the 6 outlets. The state (p) of the system implies that altogether p of the N sources are engaged (and altogether p of the $n = 6$ devices). Depending on how the p occupations are distributed among the inlet groups and among the n outgoing circuits, different numbers of blocked inlet groups are obtained (if here $p \geq 3$). Equation (TGD 1.11) here relates to the total number of occupations in the system. It cannot be applied to a part of the system.

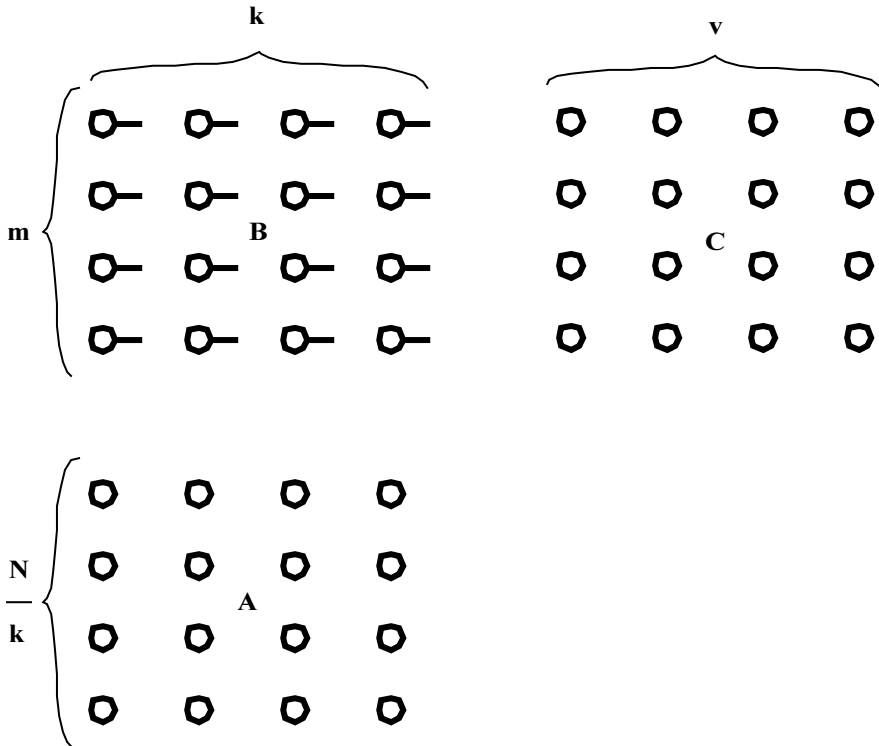


Figure TGD 1/3 : Two-stage link system

In a link system, the inlets are also divided into groups, inlet columns, in Figure TGD 1/3 with N/k inlets per column. For a loss system, the number of occupations is always equal in every stage. Therefore, a total of p engaged sources also implies a total of p engaged links and a total of p engaged outlets.

Equation (TGD 1.11) holds for a link system only if by p is meant the total number of engaged sources in the system. It can however be shown that (TGD 1.11) can be approximately applied also to a part of a link system.

2. Application on different cases

Termination

In the sequel, we shall assume exclusively an exponential distribution of holding times. If the duration of the individual occupation is described by:

$$f(t) = \frac{1}{s} e^{-\frac{t}{s}} \quad (\text{TGD 2.1})$$

where $s = \int_{t=0}^{\infty} t \cdot f(t) \cdot dt = \text{mean holding time.}$

The probability of termination of an occupation in $(t, t + \Delta t)$ is:

$$P(\Delta t) = 1 - e^{-\frac{\Delta t}{s}} \quad (\text{TGD 2.2})$$

and that it does not terminate

$$Q(\Delta t) = 1 - P(\Delta t) = e^{-\frac{\Delta t}{s}} \quad (\text{TGD 2.3})$$

If there are p independent occupations with the same holding time distribution, the probability that exactly v of them terminate in $(t, t + \Delta t)$ is:

$$P_v(\Delta t) = \binom{p}{v} \cdot \left(1 - e^{-\frac{\Delta t}{s}}\right)^v \cdot e^{-\frac{\Delta t}{s} \cdot (p-v)} \quad (\text{TGD 2.4})$$

For $v = 0$

$$P_0(\Delta t) = e^{-\frac{\Delta t}{s} \cdot p} \quad (\text{TGD 2.5})$$

The probability that at least one occupation terminates is then:

$$Q_0(\Delta t) = 1 - P_0(\Delta t) = 1 - e^{-p \cdot \frac{\Delta t}{s}} \quad (\text{TGD 2.6})$$

For small values of Δt :

$$Q_0(\Delta t) \approx p \cdot \frac{\Delta t}{s} \quad (\text{TGD 2.6a})$$

The expected number of terminations in $(t, t + \Delta t)$ is:

$$\mathcal{E}(v) = \sum_{v=0}^p v \cdot \binom{p}{v} \cdot \left(1 - e^{-\frac{\Delta t}{s}}\right)^v \cdot e^{-\frac{\Delta t}{s} \cdot (p-v)} \quad (\text{TGD 2.7})$$

$$\mathcal{E}(v) = p \cdot \left(1 - e^{-\frac{\Delta t}{s}}\right)$$

which for small Δt can be written:

$$\mathcal{E}(v) \approx p \cdot \frac{\Delta t}{s} \quad (\text{TGD 2.7a})$$

We may therefore say that the termination intensity for p exponentially distributed and independent occupations is

$$\boxed{\mu_p = \frac{p}{s}} \quad (\text{TGD 2.8})$$

(TGD 2.8) will be used in the sequel, so that (TGD 1.13) can be written:

$$\left. \begin{aligned} [p] &= \frac{\prod_{\rho=0}^{p-1} \lambda_\rho \cdot s^p}{p!} \cdot [0] \\ \sum_p [p] &= 1 \end{aligned} \right\} \quad (\text{TGD 2.8a})$$

It is sometimes convenient to use s as time unit (s = 1).

Increase:

The assumption for λ_p is

$$\boxed{\lambda_p = y(p) \cdot W(p)} \quad (\text{TGD 2.9})$$

where $y(p)$ = call intensity of the sources at state (p)
 $W(p)$ = ability of the system to accept a new occupation at state (p)

Meaning of W(p)

$W(p) = 1$ means: A new call can always seize a device in the system;

$W(p) = 0$ means: The system cannot accept a new call;

$0 < W(p) < 1$ means: Only certain calls can be accepted by the system.

For a full availability group ($N > n$) in a loss system

$$\begin{aligned} W(p) &= 1 \text{ for } 0 \leq p < n \\ W(p) &= 0 \text{ for } p = n \end{aligned} \quad (\text{TGD 2.10})$$

in a delay system

$$W(p) = 1 \text{ for } 0 \leq p \leq N \quad (\text{TGD 2.11})$$

as unsuccessful calls are allowed to wait.

For a grading or a link system in a loss system

$W(p) = 1$ for $p <$ given limit. Calls can be accepted from all inlet groups.

$0 < W(p) < 1$ for such values of p that the system can only accept calls from certain inlet groups or inlet columns to certain routes.

$W(p) = 0$ total congestion prevails, no calls can be accepted.

Call Intensity y(p)

Consider a single source. Assume that:

$$f(t) = \beta \cdot e^{-\beta t} \tag{TGD 2.12}$$

describes the distribution for the time elapsing from the moment the source becomes free to the moment it calls again, i.e. f(t) is the distribution of call intervals.

The conditional probability that the source makes a new call in the interval (t, t + Δt) is then:

$$P(\Delta t) = 1 - e^{-\beta \cdot \Delta t} \tag{TGD 2.13}$$

For small Δt

$$P(\Delta t) \approx \beta \cdot \Delta t \tag{TGD 2.13a}$$

Consider x independent, free sources with the same distribution of call intervals, f(t). The probability that v of x sources make calls in (t, t + Δt) is then:

$$P_v(\Delta t) = \binom{x}{v} \cdot (1 - e^{-\beta \cdot \Delta t})^v \cdot e^{-\beta \cdot \Delta t(x-v)} \tag{TGD 2.14}$$

The expected number of calls in the interval is:

$$\mathcal{E}(v) = x \cdot (1 - e^{-\beta \cdot \Delta t}) \tag{TGD 2.15}$$

For small Δt, this can be written:

$$\mathcal{E}(v) = x \cdot \beta \cdot \Delta t \tag{TGD 2.15a}$$

The call intensity with x free sources is consequently:

$$y_x = x \cdot \beta \tag{TGD 2.16}$$

For y(p), the following general assumptions are now made:

$y(p) = (N - p) \cdot \beta$ (BE) Bernoulli, Engset	(TGD 2.17)
$y(p) = y$ (E) Poisson, Erlang	
$y(p) = \beta \cdot (\gamma + p)$ (NB) Negative binomial type	

The assumptions result in traffic distributions of known types. (BE) is a distribution in which the call intensity decreases with increased number of occupations, (E) is independent of the number of occupations, and (NB) has a call intensity increasing with the number of occupations.

One assumption (TGD 2.17) can be made to include another, viz :

BE → E	if N → α	β → 0,	N · β → y	
BE → NB	if N < 0,	β < 0,	N · β → aγ,	-β → a
NB → BE	if a < 0,	γ < 0,	aγ → Nβ	-a → β
NB → E	if a → 0,	γ → ∞	a → y	

Insertion of (TGD 2.17) in (TGD 2.9) and (TGD 2.8a) then results in different distributions, which are dealt with in the following sections for the full availability group in loss systems. But before that, some general concepts for a traffic distribution will be defined.

3. Some general concepts and definitions

Time Congestion is denoted by E.

E = THE PROPORTION OF THE TIME DURING WHICH CONGESTION PREVAILS = PROBABILITY THAT ALL AVAILABLE DEVICES ARE ENGAGED	(TGD 3.1)
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For delay systems, E = probability that all devices are occupied independent of if calls are waiting or not.

Call Congestion is denoted by B.

B = THE PROPORTION OF UNSUCCESSFUL CALLS = PROBABILITY OF AN UNSUCCESSFUL CALL	(TGD 3.2)
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$$E = \sum_{p \geq n} [p] \quad , \text{ if } p \geq n \text{ denotes all states (p) when congestion prevails.} \quad (\text{TGD 3.3})$$

$$B = \frac{\sum_{p \geq n} [p] \cdot y(p)}{\sum_{\text{all } p} [p] \cdot y(p)} \quad = \text{expected number of calls when congestion prevails} \\ \text{divided by the total number of expected calls.} \quad (\text{TGD 3.4}) \\ \text{Calculated per time unit}$$

For delay systems, B = P (t>0) = the probability of having to wait.

For gradings and link systems, the above summation $p \geq n$ must be regarded as symbolical.

Traffic carried is here denoted by A^1 = mean number of simultaneous occupations.

Measurable! $A^1 = \sum_{p=1}^n p [p] \quad n = \text{max. number of simultaneous occupations} \quad (\text{TGD 3.5})$

As: $\frac{p}{s} \cdot [p] = \lambda_{p-1} \cdot [p-1] \quad (\text{TGD 3.6})$

A^1 can also be written:

$$A^1 = \sum_{p=1}^n s \cdot \lambda_{p-1} \cdot [p-1] = s \sum_{p=0}^{n-1} \lambda_p \cdot [p] \quad (\text{TGD 3.5a})$$

Traffic Offered is denoted by A

$$A = \sum_{\text{all } p} s \cdot y(p) \cdot [p] \quad (\text{TGD 3.7})$$

= mean number of simultaneous occupations offered to the system (whether accepted or not).

**A IS DEPENDENT EXCLUSIVELY ON THE THEORETICAL MODEL USED.
CANNOT BE MEASURED!**

Difference between A and A^l:

Write
$$A = \sum_{p=0}^r s \cdot y(p) \cdot [p] \quad r = \text{max. of } p \quad (\text{TGD 3.7})$$

$$A^l = \sum_{p=0}^{n-1} s \cdot W(p) \cdot y(p) \cdot [p] \quad (\text{TGD 3.5b})$$

where n = max. number of simultaneous occupations, ≤ r.

This gives:
$$\Delta A = A - A^l = \sum_{p=0}^{n-1} s \cdot (1 - W(p)) \cdot y(p) \cdot [p] + \sum_{p=n}^r s \cdot y(p) \cdot [p] \quad (\text{TGD 3.8})$$

or
$$\Delta A = A - A^l = \sum_{p=n}^r s \cdot y(p) \cdot [p] \text{ if } W(p) = 1 \text{ for } p < n$$

We can also write:

$$\Delta A = A - A^l = s \cdot \sum_{p=0}^r y(p) \cdot (1 - W(p)) \cdot [p] \quad (\text{TGD 3.8a})$$

$\Delta A > 0$ only if $W(p) < 1$ occurs for $0 \leq p \leq r$

This means that:

$A > A^l$ for loss system $A = A^l$ for a delay system (*)	(TGD 3.9)
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(*) if all calls wait until served.

Load on the v:th device

Denoted a_v

Different expressions for sequential hunting and random hunting, for loss systems and delay systems.

The improvement factor is denoted F.

If a system is extended from n to n + Δn devices without change of the traffic offered, F is defined as:

$$F = A^l(n + \Delta n) - A^l(n) \quad (\text{TGD 3.10})$$

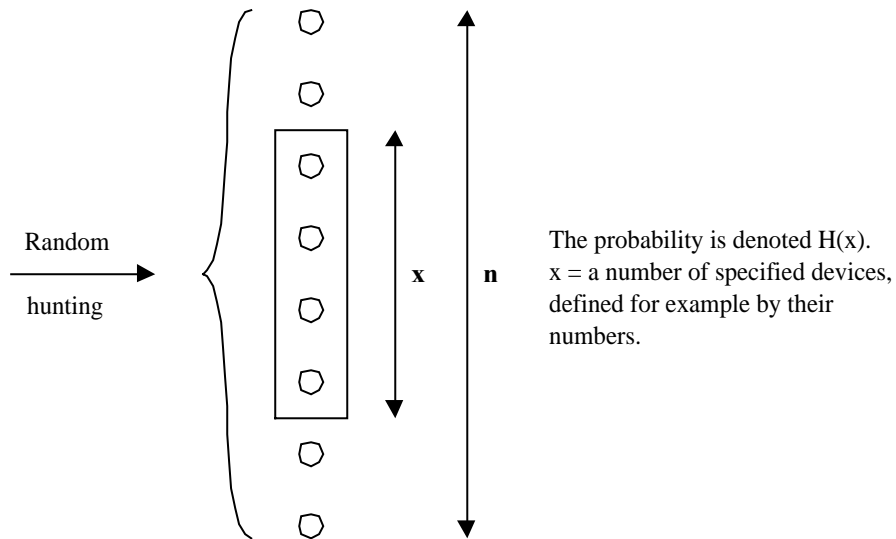
i.e. the improvement factor is the increase in the traffic handled for an increase from n to n + Δn devices.

Usually $\Delta n = 1$.

NOTE that a system can often be extended in more than one way. F must then be considered as multidimensional.

Probability of x specified devices engaged

Random hunting = same load on all devices = all combinations with same number of engaged devices equally probable.



$$H(x) = \sum_{p=x}^n [p] \cdot \frac{\binom{x}{x} \cdot \binom{n-x}{p-x}}{\binom{n}{p}} + \sum_{p>n} [p] \tag{TGD 3.11}$$

The second sum is = 0 for a loss system, but > 0 for a delay system.

H(x) should be applied with great caution to gradings and link systems, as all n devices do not always carry the same load and as all combinations of p engaged devices are not always equally probable.

With SEQUENTIAL HUNTING, the expressions for H(x) are very complicated.

Duration of state (p)

The system remains in (p) either until an occupation terminates or a new call occurs. The intensities of change are μ_p and λ_p .

The probability that (p) persists after time t is:

$$p_p (> t) = e^{-(\lambda_p + \mu_p) \cdot t} \tag{TGD 3.12}$$

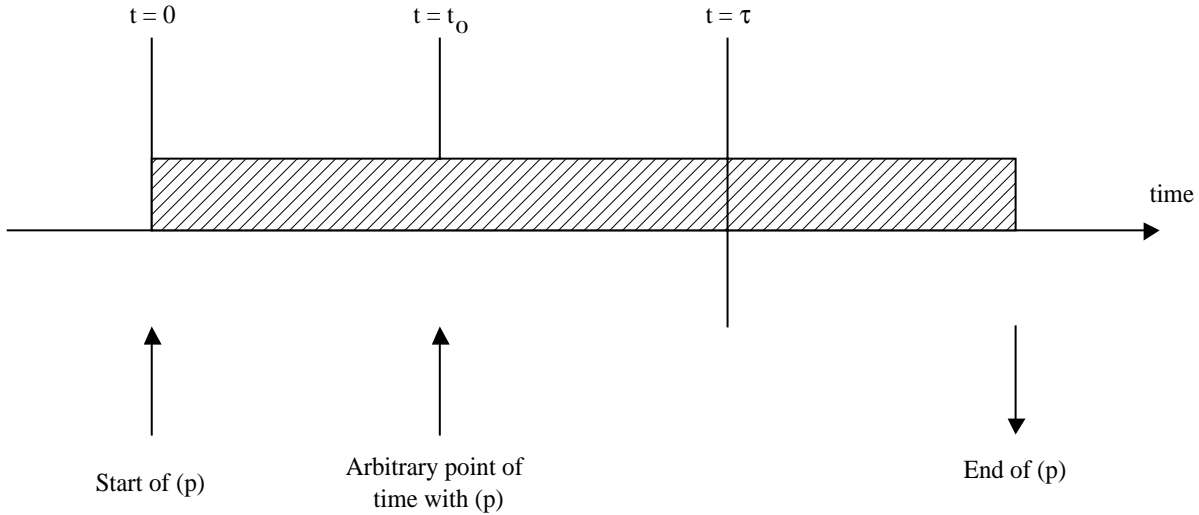
The frequency function for the duration of (p) is:

$$f_p (t) = (\lambda_p + \mu_p) \cdot e^{-(\lambda_p + \mu_p) \cdot t} \tag{TGD 3.13}$$

Mean duration:

$$\bar{t}_p = \frac{1}{\lambda_p + \mu_p} \tag{TGD 3.14}$$

NOTE the difference between total duration and remaining duration



The total duration of (p) is described by:

$$f_p(t) = (\lambda_p + \mu_p) \cdot e^{-(\lambda_p + \mu_p) \cdot t} \quad \text{(TGD 3.15)}$$

The probability that (p) exists still at τ is:

$$p_p(t > \tau) = e^{-(\lambda_p + \mu_p) \cdot \tau} \quad \text{(TGD 3.16)}$$

The probability that state (p) persists at τ when it is known that state (p) existed at $t = t_0$:

$$P(> \tau - t_0) = P(t > \tau | t > t_0) = \frac{P(t > \tau)}{P(t > t_0)}$$

$$P(> \tau - t_0) = \frac{e^{-(\lambda_p + \mu_p) \cdot \tau}}{e^{-(\lambda_p + \mu_p) \cdot t_0}} = e^{-(\lambda_p + \mu_p) \cdot (\tau - t_0)}$$

Thus: $P(> \tau - t_0) = P_p(t > \tau - t_0)$ (TGD 3.17)

i.e. the probability that (p) lasts a time θ is equal to the probability that (p) has a remaining duration θ calculated from an arbitrary point of time when the state of the system is (p). APPLIES ONLY TO EXPONENTIAL DISTRIBUTION!

Convergency condition

In order that a traffic process may attain statistical equilibrium and remain in equilibrium, the following condition must be fulfilled:

$\sum_{all \ p} [p] \cdot (\lambda_p + \mu_p) = finite$	(TGD 3.18)
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This condition implies that the mean number of changes of (p) per time unit must be finite.

4. Example of a simple state process in equilibrium

If a group of circuits is considered during a certain time interval, the number of occupations may change as described in Figure 12/1.

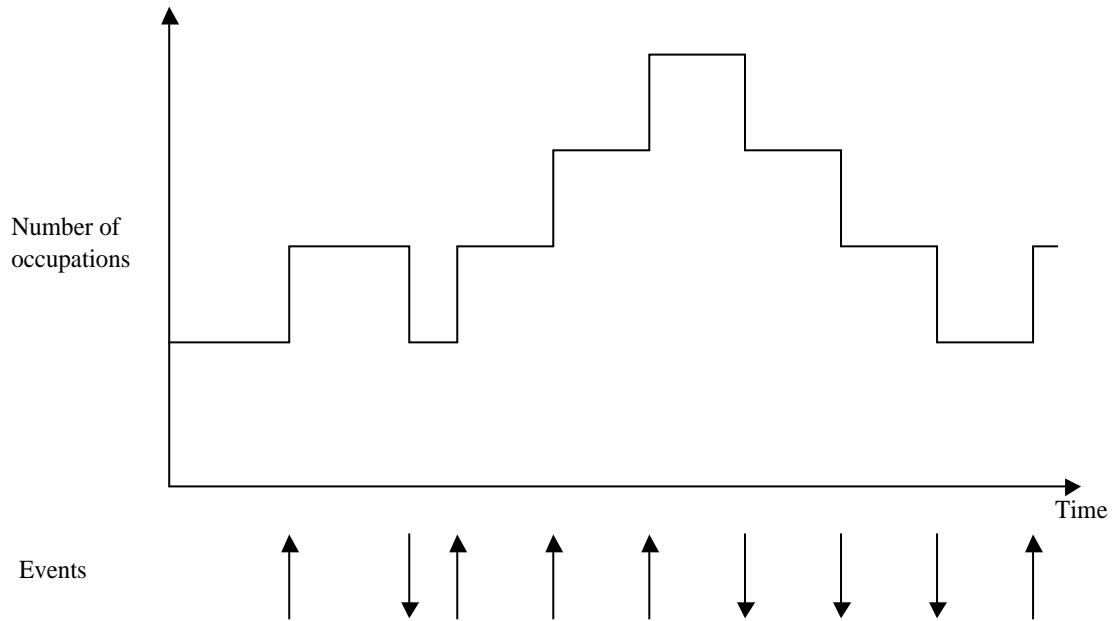


FIGURE TGD 4/1

Number of occupations changing with time, determined by occurring events.
 (Call ↑, termination ↓)

The process shown in Figure TGD 4/1 may or may not be in equilibrium. (Actually, too few events are shown in the Figure to allow any conclusions in that respect.) However, to say that a traffic process is in statistical equilibrium, we must assume that:

- the process is neither increasing nor decreasing in the long run;
- the process started so long ago that how it started is now irrelevant to its present variations.

Deduction of state probabilities

To illustrate how the state probabilities for a traffic process can be deducted, we make here the following assumptions:

- full availability group with n devices;
- call intensity y when $p < n$ and zero when $p = n$, (i.e. the call intensity is independent of the number of occupations in the group, and rejected calls do not change the call intensity => no repeated attempts!).
- the holding times are exponentially distributed with mean h .
- the process is in statistical equilibrium.

The assumptions apply for a full availability group of the Poisson-Erlang type (cf TGD 2.17).

Since statistical equilibrium implies:

INCREASE = DECREASE

in the long run, (TGD.1.11) applies:

$$\lambda_{p-1}(p-1) = \mu_p(p) \tag{TGD 1.11}$$

where $\lambda_{p-1} = W(p-1) \cdot y$ (TGD 4.1)

$$\mu_p = \frac{p}{h} \quad \text{(TGD 4.2)}$$

Since a full availability group is considered, we can write:

$$W(p) = 1 \quad \text{for } 0 \leq p < n$$

$$W(p) = 0 \quad \text{for } p = n$$

Then, we have: $y \cdot (p-1) = \frac{p}{h} \cdot (p)$ (TGD 4.3)

for $p = 1, 2, 3, \dots, n$

Introducing : $A = yh$, we can write:

$$A [p-1] = p \cdot [p] \quad \text{(TGD 4.4)}$$

which means: $A(0) = (1)$

$$A(1) = 2(2)$$

$$A(2) = 3(3) \quad \text{(TGD 4.4a)}$$

$$A(p-1) = p \cdot (p)$$

$$A(n-1) = n \cdot (n)$$

It follows that any state probability can be expressed in its neighbour state probability, and we can therefore make recursions from any (p) to the state (0), i.e.:

$$(1) = A \cdot (0)$$

$$(2) = \frac{A}{2} \cdot A \cdot (0)$$

$$(3) = \frac{A}{3} \cdot \frac{A}{2} \cdot \frac{A}{1} \cdot (0)$$

etc., i.e.:

$$(p) = \frac{A^p}{p!} \cdot (0) \quad \text{(TGD 4.5)}$$

where $p! = p \cdot (p-1) \cdot (p-2) \dots 2 \cdot 1$

According to the definition of probabilities,

$$\sum_{p=0}^n (p) = 1 \quad \text{(TGD 4.6)}$$

i.e. the sum of all possible state probabilities is equal to unity. Therefore:

$$\sum_{p=0}^n \frac{A^p}{p!} \cdot (0) = 1 \quad \text{(TGD 4.7)}$$

if we define $p! = 0$ for $p = 0$,

or:
$$(0) \cdot \left(1 + A + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} \right) = 1$$

We can therefore write:

$$(p) = \frac{\frac{A^p}{p!}}{\sum_{v=0}^n \frac{A^v}{v!}} \quad \text{(TGD 4.8)}$$

where $p = n$ gives Erlang's First Formula.

Classical deduction of the Erlang distribution

Same assumptions apply as above. We consider the state process within a short interval $(t, t + \Delta t)$ and describe what may happen in this interval. For an arbitrary state (p) , we calculate the probability that the process will have exact p occupations at $t + \Delta t$.

These possibilities appear:

- There were already p occupations at t and no new calls or terminations occurred in the interval;
- there were less than p occupations at t and a pertinent number of calls and no terminations occurred;
- there were more than p occupations at t but a pertinent number of terminations, but no calls occurred in the interval;
- both calls and terminations occurred in the interval but in such a way that there were exactly p occupations at $t + \Delta t$.

We can now write the following expression for the probability of p occupations at $t + \Delta t$.

$$\begin{aligned} [P]_{t+\Delta t} &= [p]_t \{ 1 - A\Delta t - p\Delta t \} + && \text{(no event)} \\ &+ [p-1]_t \cdot A \cdot \Delta t + && \text{(no event)} \\ &+ [p+1]_t \cdot (p+1) \Delta t && \text{(no event)} \\ &+ [x]_t \cdot && \text{Probability of more than one event.} \end{aligned}$$

(TGD 4.9)

$0 \leq x \leq n$

To understand the last term in the equation, we can consider the probability that the state is changed from $p-2$ to p in $(t, t + \Delta t)$. The probability for this is:

$$[p-2] \cdot A \Delta t \cdot A \Delta t = [p-2] A^2 \cdot \Delta t^2$$

In fact, if z events occur, its probability is proportional to $(\Delta t)^z$. Therefore, if we let Δt^z become small enough, all cases with more than one event in $(t, t + \Delta t)$ can be neglected. We can then write (TGD 4.9) as:

$$[p]_{t+\Delta t} - [p]_t = - [p]_t \cdot (A+p) \Delta t + [p-1]_t A \cdot \Delta t + [p+1]_t (p+1) \Delta t$$

or
$$\frac{[p]_{t+\Delta t} - [p]_t}{\Delta t} = - [p]_t \cdot (A+p) + [p-1]_t \cdot A + [p+1]_t \cdot (p+1) \quad \text{(TGD 4.10)}$$

If we let $\Delta t \rightarrow 0$, the left hand member of (TGD 4.10) becomes a derivative

$$\frac{[p]_{t+\Delta t} - [p]_t}{\Delta t} \rightarrow \frac{d[p]_t}{dt}$$

Equilibrium assumes that the state probabilities (p) do not change with time, i.e.: $\frac{d(p)_t}{dt} = 0$

The equilibrium assumption means that the state probabilities (p)_i remain constant independent of time, t. We can, therefore, write them simply (p).

For $\frac{d[p]_t}{dt} = 0$, we can now write (2.5.10) as follows:

$$A [p] - (p + 1) [p - 1] = A [p - 1] - p [p]$$

For p = 0, [p-1] = 0, and therefore A [0] - [1] = 0

For p = 1 A [1] - 2 [2] = A [0] - [1] = 0

We consequently arrive at: $A [p - 1] = p \cdot [p]$ (TGD 4.4)

as given above.

Remark: The expression

$$\frac{d[p]}{dt} = -[p](A + p) + [p - 1] \cdot A + p + 1 \quad (\text{TGD 4.11})$$

provides possibilities to study the process before it has reached its equilibrium state, such as starting conditions.