**Dimensioning & Optimization**

**of Junction Network**

Mr. T. Fried, ITU

**UNION INTERNATIONALE DES TELECOMMUNICATIONS**
**INTERNATIONAL TELECOMMUNICATION UNION**
**UNION INTERNACIONAL DE TELECOMUNICACIONES**

**Contents :**

This document consists of the following sub-chapters from the documentation of the PLANITU software

## 2.4 TRAFFICS

Before dimensioning/optimizing the circuit groups between the exchanges in the network, a ***traffic matrix*** between these exchanges has to be set up. This chapter describes the calculation method for
the types of network considered, ie

- local networks ( rural, metropolitan )
- long distance networks ( national, regional )

### 2.4.1 Traffic zones and subscribers

For local networks, traffics calculations are based on subscribers/exchange, subscriber's categories and traffic zones.

**Assumptions :**

- the area under consideration has been divided into ***traffic zones***; the subscribers belonging to such a zone are assumed to have ***uniform traffic properties,*** such as traffic originated and terminated per subscriber, and traffic dispersion to other zones;
- the number of subscribers of any zone,T, are known for any given exchange, E; they may have been defined in the input data, or calculated in the previous boundary optimization: **NSUB(E,T);**
- the total number of subscribers belonging to any traffic zone,T, is known; this has been calculated after reading the input data concerning ***zone definition*** and *subscriber distribution*: $\mathrm{SUBTZ}_{\mathrm{T}}$
- the total traffic from any traffic zone, T, to any other traffic zone, U, is known from input data : $\mathrm{A}_{\mathrm{TU}}$

The ***specific traffic interest*** between one subscriber in traffic zone T and one subscriber in traffic zone U can then be expressed as

$$a(T,U) = \frac{A_{TU}}{SUBTZ_T \cdot SUBTZ_U}$$

Finally, the traffic from any exchange, E, to any other exchange, F, can now be written as

$$Traffic(E,F) = \sum_{T,U} [NSUB(E,T) \cdot NSUB(F,U) \cdot a(T,U)]$$

### 2.4.2 Traffic matrix

For long distance networks, or for local networks where the boundaries are nor subject to investigation by the program, a traffic matrix is given defining the traffic interest between any pair of exchanges. For sub-exchanges, remote subscriber units, etc, total incoming and outgoing traffic can be specified to enable dimensioning of routes to/from the parent exchange.

The traffics given in this matrix are busy hour traffics.

In the case of ***non-coincident busy hours*** the traffic matrix contains the total daily traffic volume in erlanghours, and information about the relevant traffic profile has to be supplied. The traffic value for any given time, t, is then found by multiplying the total traffic by the traffic profile value for t, ie

$$\mathrm{Traffic(E,F,t)} \;=\; \mathrm{A(E,F) \cdot PROF(E,F,t)}$$

There is obviously no need to define a specific traffic profile for each particular traffic case. Exchanges can be grouped into categories, and the traffic profiles defined between categories rather than between exchanges.

For local and national networks this grouping of exchanges would probably depend on the percentage of subscriber categories, such as residential, business, public services, PABXes, etc; as the telephone habits for each of these categories can be measured or estimated, the profile for a given "mix" of categories can be found.

For international networks, the profiles would rather depend on the time difference between the countries involved, and on any special relations between the countries.

## 2.5 DIMENSIONING/OPTIMIZATION OF ROUTES
### CALCULATION OF OVERFLOW TRAFFICS

This chapter deals with the task of providing circuits between the various exchanges in the network in such a way that the overall cost of the network is minimised, taking into account

- the grade of service desired;
- the properties of traffic offered;
- the technical properties of the switching equipment;
- the costs of the switching and transmission equipment.

This chapter describes the mathematical solution of dimensioning and optimization, while the remaining numerical problems will be dealt with in chapter 3.5.

The following notations will be used throughout this chapter:

N = number of circuits on a route
k = availability, or number of outlets to a route per group selector unit
A = mean = variance of a Poisson-type traffic
M = mean of traffic offered to an alternative route
V = variance of ditto
m = mean of overflow traffic from a route
v = variance of ditto
B = m/M = congestion on a route.

### 2.5.1 Congestion Theory Full Availability

**Primary routes**

Assuming Poisson-type offered traffic, the mean and variance of overflow traffic from a direct route are given by the Wilkinson formulae:

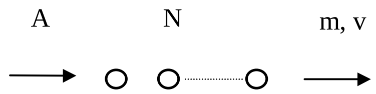
$$m = A \cdot E_N(A)$$

$$v = m \cdot \left(1 - m + \frac{A}{N + 1 - A + m}\right)$$

where $E_N(A)$ denotes the Erlang loss formula

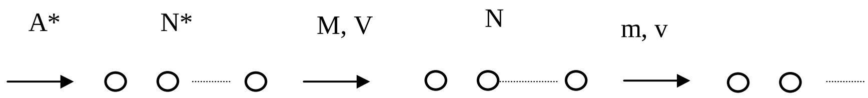
$$E_N(A) = \frac{\dfrac{A^N}{N!}}{\sum_{i=0}^{N} \dfrac{A^i}{i!}}$$

The following figure illustrates the Wilkinson model for primary routes :



**Alternative routes**

The offered traffic of mean M and variance V is substituted by the traffic of equal mean and variance overflowing from a fictive, fully available group offered Poisson-type traffic :




The parameters $N^*$ and $A^*$ of this "equivalent group" can be determined from the equation system

$$M = A^* \cdot E_{N^*}(A^*)$$

$$V = M \cdot \left(1 - M + \frac{A^*}{N^* + 1 - A^* + M}\right)$$

See chapter 3.7 for method of determining $N^*$ and $A^*$ .

The mean and variance of the overflow traffic from the alternative route are then estimated as

$$m = A^* \cdot E_{N+N^*}(A^*)$$

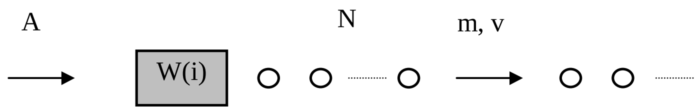
$$v = m \cdot \left(1 - m + \frac{A^*}{N + N^* + 1 - A^* + m}\right)$$

### 2.5.2 Congestion Theory Restricted Availability

The problem of congestion on routes with restricted availability are solved by introducing the *loss function*, $W_i$, denoting the conditional probability of a call overflowing from the route, given that precisely ***i*** circuits are occupied when the call arrives. Thus $W_i$, is to describe the main properties of the actual grading and/or link system. The moments of the overflow traffic are derived from equations of state for an *arbitrary* probability function $W_i$,. An example of such a function is shown in Chapter 3.8.

**Primary routes**

The calculation scheme for primary routes is illustrated in the following figure; the "box" in front of the route indicates restricted availability.




The overflow traffic mean and variance can be written as

$$m = A \cdot \sum_{i=0}^{N} W_i \cdot P(i)$$

$$v = m - m^2 + A \cdot \sum_{i=0}^{N} W_i \cdot Q(i)$$

where P(i) can be determined from the relations

$$P(i)\cdot A\cdot(1-W_i)=P(i+1)\cdot(i+1) \qquad \text{for i = 0,1,2,....,N-1}$$

$$\sum_{i=0}^{N} P(i)=1$$

and Q(i) can be determined from the relations

$$Q(N-1)\cdot A\cdot(1-W_{N-1})=Q(N)\cdot(N+1)-P(N)\cdot A\cdot W_N$$

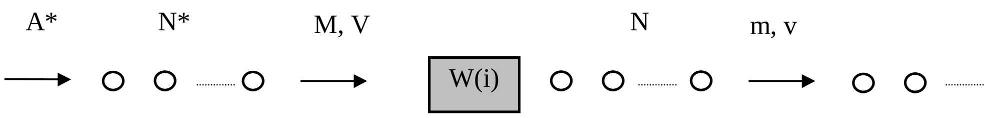
$$Q(i)\cdot A\cdot(1-W_i)=Q(i+1)\cdot[i+2+A\cdot(1-W_{i+1})]-$$
$$-Q(i+2)\cdot(i+2)-P(i+1)\cdot A\cdot W_{i+1} \qquad \text{for i = N-2,N-3,....,0}$$

$$\sum_{i=0}^{N} Q(i)=m$$

See chapter 3.5 for numerical considerations of these calculations.

**Alternative routes**

An "equivalent group",$(N^*, A^*)$, corresponding to the given mean and variance of traffic offered is determined as shown in the case of full availability. Thus the substitute scheme in the following figure is obtained:




The "exact" solution given by Wallström is, unfortunately, too complicated for practical employment in network planning programs.

To calculate the required moments of overflow traffic we make use of *approximate* solutions, which have been compared with both the "exact" solution, and extensive simulations, and found to be sufficiently good. Two approximations are described below.

**Approximation I** : The required moments are constructed from the corresponding moments of the simpler cases described above. Thus,we can write

$$m=m_2+m_3-m_1$$

$$\frac{v}{m}=\frac{v_2}{m_2}+\frac{v_3}{m_3}-\frac{v_1}{m_1}$$

where the subscripts indicate the simpler cases, that is

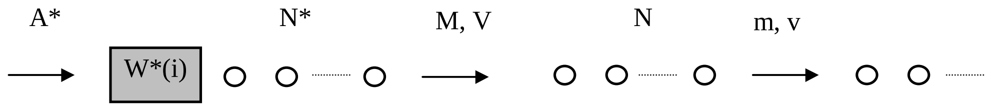
1 : Full availability, primary routes
2 : Full availability, alternative routes
3 : Restricted availability, primary routes

For dimensioning of routes for a given grade of service, the corresponding approximation would be :

$$N=N_2+N_3-N_1$$

The approximation for N is acceptable only for small values of B, (B < 0.05), while the approximation for m and v is best for higher values of B (B > 0.15).

**Approximation II** : Here the calculation scheme shown above is changed slightly to the following figure :




where $W_i^* = 0$ for $i = 0,1,2,\ldots,N^*-1$

$W_i^* = W_{i-N^*}$ for $i = N^*, N^*+1,\ldots,N^*+N$

and $W_i$ is calculated in the usual way.

Now $P(i)$ , $Q(i)$, m and v can be calculated as previously described, using ($N^* + N$) and $A^*$ instead of A and N.

## 2.5.3 Dimensioning

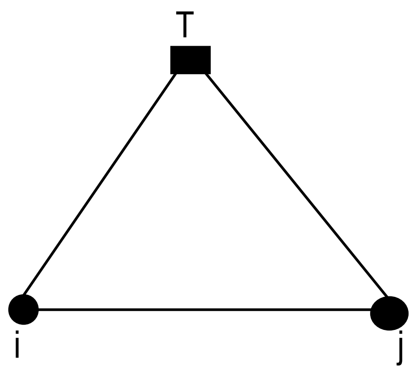
Routes between exchanges can now be dimensioned based on the formulae or methods given above.

As the congestion B(N) steadily *decreases* for *increasing* N, starting with B(N) = 1 for N = 0, and $B(N) \rightarrow 0$ for $N \rightarrow \infty$, the method that suggests itself is to start with a suitable value for N, and then increasing N in suitable steps until the resulting B(N) becomes smaller than, or equal to, a predescribed value, B. For details of calculation, considering the numerical problems that may occur for large values of N, see chapter 3.5.

### 2.5.4 Optimisation of circuits on high-usage routes

It is practically impossible to optimise all the routes in an alternative routing network simultaneously, so we have to resort to sub-optimisation, ie treating only one particular route, assuming the rest of the network already optimised and dimensioned. Starting from some approximate solution, we will then iteratively arrive at the cost optimum by treating every route individually, making use of previous results for the other routes in the network.

Consider the simple configuration of three exchanges shown below :




There are 2 exchanges, i and j, on the lowest level of an assumed hierarchical network structure, and another exchange, T, called the transit or tandem exchange, on a higher level.

Considering the traffic case from $i \rightarrow j$ , there are 3 possibilities of routing the traffic, ie

- all traffic is carried on the route from i to j; we denote this case as **D** for Direct routing
- all traffic is carried through the tandem exchange, T ; we denote this case as **T**

- part of the traffic is carried on the route i -> j , and the rest of the traffic overflows to the routes $i \to T \to j$. We denote this case as **H** for High-usage route.

It is the third case that interests us here, in particular the question what proportion of the traffic should be carried on the direct and overflow routes, resp. As we are interested in **optimization**, the question then will be "what proportion will result in the least cost of satisfying the traffic demand i to j for given costs and grade of service?".

For the simple triangular routing pattern shown above the following data can then be assumed to be known :

| | |
|---|---|
| M,V | mean and variance of offered traffic to route $i \to j$ |
| $M_{10}, V_{10}$ | mean and variance of traffic offered to route $i \to T$ , not including the overflow traffic from route $i \to j$ |
| $M_{20}, V_{20}$ | ditto for route $T \to j$ |
| $c, c_1, c_2$ | cost of one additional circuit on route $i \to j$ ,$i \to T$ , and $T \to j$ , resp. |

If we denote the mean and variance of the overflow traffic from route $i \to j$ as m(N) and v(N), where N denotes the number of circuits on route $i \to j$ ,we get the total traffic offered to the overflow routes as

$$M_1 = M_{10} + m(N) \qquad M_2 = M_{20} + m(N)$$

$$V_1 = V_{10} + v(N) \qquad V_2 = V_{20} + v(N)$$

We wish to determine N so that the resulting cost is minimised, ie we have to minimise the cost function

$$C(N) = c \cdot N + c_1 \cdot N_1 + c_2 \cdot N_2$$

with regard to N. $N_1$ and $N_2$ are the circuits on the overflow routes.

As the grade of service for the overflow routes is assumed to be known, $N_1$ and $N_2$ are functions of N. To find the minimum of the cost function, we should equate its derivative, with respect to N, to zero.

We get

$$c + c_1 \cdot \frac{\partial N_1}{\partial N} + c_2 \cdot \frac{\partial N_2}{\partial N} = 0$$

From this relation the optimal value of N can then be found as follows:

For the first overflow route we know that, for a given grade of service,

$$N_1 = N_1(M_1, V_1)$$

or, alternatively, for $\Theta_1 = \frac{V_1}{M_1}$,

$$N_1 = N_1(M_1, \Theta_1)$$

We can then write

$$\frac{\partial N_1}{\partial N} = \frac{\partial N_1}{\partial M_1} \cdot \frac{\partial M_1}{\partial N} + \frac{\partial N_1}{\partial \Theta_1} \cdot \frac{\partial \Theta_1}{\partial N}$$

From

$$M_1 = M_{10} + m \qquad \text{and} \qquad \Theta_1 = \frac{V_{10} + v}{M_{10} + m}$$

we find

$$\frac{\partial M_1}{\partial N} = \frac{\partial m}{\partial N} \qquad \text{and} \qquad \frac{\partial \Theta_1}{\partial N} = \frac{\frac{\partial v}{\partial N} - \Theta_1 \cdot \frac{\partial m}{\partial N}}{M_1}$$

As the second expression is usually quite small, we then get approximately

$$\frac{\partial N_1}{\partial N} \approx \frac{\partial N_1}{\partial M_1} \cdot \frac{\partial m}{\partial N} \qquad \text{and} \qquad \frac{\partial N_2}{\partial N} \approx \frac{\partial N_2}{\partial M_2} \cdot \frac{\partial m}{\partial N}$$

Finally, by inserting these derivatives in the derivative of the cost function, we obtain

$$-\frac{\partial m}{\partial N} = \frac{c}{c_1 \cdot \frac{\partial N_1}{\partial M_1} + c_2 \cdot \frac{\partial N_2}{\partial M_2}}$$

The derivatives $\frac{\partial N_1}{\partial M_1}$ and $\frac{\partial N_2}{\partial M_2}$ can be calculated a shown in Chapter 3.6, and are *constants* during the sub-optimisation of the route under consideration; so are c, $c_1$ and $c_2$. The optimisation problem is thus reduced to finding a value for N so that

$$-\frac{\partial m}{\partial N} = \text{constant}$$

Although this expression easily yields the optimal N for routes with full availability, for routes with restricted availability ( eg graded routes ) derivatives cannot be found easily. We will therefore ....

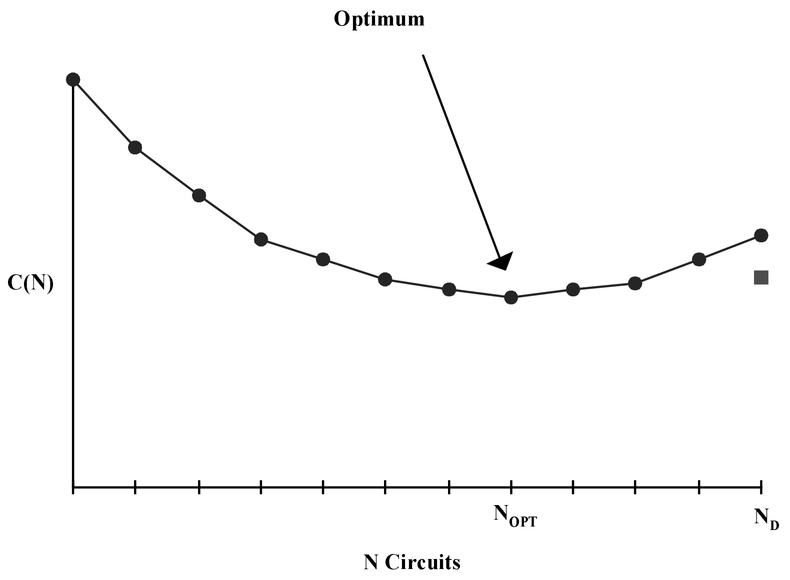
The divisor in the right-hand part of the optimisation equation,

$$c_1 \cdot \frac{\partial N_1}{\partial M_1} + c_2 \cdot \frac{\partial N_2}{\partial M_2}$$

is obviously the cost per erlang overflowing to the tandem routes, which we will denote as $C_T$. The original cost equation can thus be re-written as

$$C(N) = c \cdot N + m(N) \cdot C_T$$

The following figure illustrates the cost expression for various values of N, for a specific offered traffic and $C_T$ :




The points marked ● correspond to the above optimisation equation, valid for N = 0,1,2,...,$N_D$, and the point ■ marked correponds to the cost for providing a direct route without overflow.

So, the obvious procedure to follow is to calculate the value of the above expression for successive values of N, starting with N=0 until reaching either $N_D$, or the lowest value of **C**(N).

Should the overflow path have more than the two routes assumed in the above triangular configuration, then there will be the corresponding number of additional $C_T$-terms.

For any route in the overflow path, it should be observed that

- if the route is final, ie without overflow possibilities, the derivative $\frac{\partial N}{\partial M}$ should be calculated for a *constant congestion value*;
- if the route does have overflow possibilities, the derivative $\frac{\partial N}{\partial M}$ should be calculated for a *constant overflow traffic mean.*

For further details please see Chapter 3.6.

**<u>Initial values</u>**

The iteration process outlined in Chapter 2.1 makes use of the traffic flows and circuit calculations of the previous iteration when optimising any particular route. In the first iteration, such information is not yet available, and some assumption about the *efficiency* of the routes on the overflow path has to be made. This can be accomplished by setting the value of the derivative to an empirically found constant before starting the first iteration, ie $\frac{\partial N}{\partial M} = 1.2$

### 2.5.5 Hierarchical networks considerations

**<u>Order of calculations</u>**

As a route can be optimised/dimensioned only when all other routes with traffic overflowing to that route have been calculated, it is important that the order in which routes are treated is strictly conforming to the hierarchical network structure, and the specified routing rules.

**<u>Service criteria</u>**

The required grade of service in the network can be determined prior to starting the calculations in one of two ways, ie

- specified grade of service for *final routes*;
- specified grade of service for *traffic cases.*

In the second case the required GOS for any final route will then have to be set in such a way that the overall congestion experienced by any traffic using that route will not be larger than the specified value. To ensure this, the GOS values have to be carefully constructed depending on the results of the circuits provided on the lower levels of the network, and the ensuing congestion values and overflow traffics. As a final route may carry partial traffics overflowing from many lower level routes, the "worst" case should determine the GOS to be applied to the final route.

Before discussing how to set the GOS-values for higher-level routes, let us first consider how the overall grade of service for a given traffic case is calculated.

The total congestion for any traffic case can be calculated approximately when all overflow traffics and circuits on the overflow paths concerned have been calculated.

On a route where

M = offerered traffic mean
V = offered traffic variance
B = congestion

an individual traffic stream, i, with

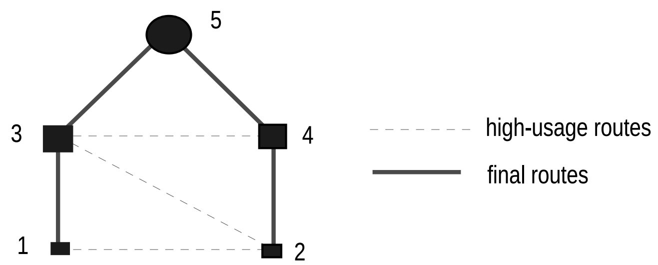
$m_i$ = offered traffic mean
$v_i$ = offered traffic variance

will experience a congestion, $B_i$ , that can be approximately expressed as

$$B_i \approx B \cdot \frac{\frac{v_i}{m_i}}{\frac{V}{M}}$$

Thus, to calculate the *total congestion* for any traffic case, the sequence of overflow routes has to be carefully examined, and the corresponding terms for congestion and overflowing traffics added and/or multiplied accordingly. The example below illustrates the method.

Assume a 3-level hierarchical network, the relevant routes of which are shown in this figure :




We are interested to find the total grade of service for the traffic case $1 \rightarrow 2$. A call offered to this route has 4 possible alternatives, ie

$1 \rightarrow 2$ $\quad$ $1 \rightarrow 3 \rightarrow 2$ $\quad$ $1 \rightarrow 3 \rightarrow 4 \rightarrow 2$ $\quad$ $1 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 2$

For any of the routes involved, the traffic offered (M,V), the congestion (B), and the overflow traffic (m,v) are assumed to be known. The overall grade of service, $T_{12}$, can then be found in this way :

$$T_{12} = B_{12} \cdot \frac{v_{12}}{m_{12}} \cdot \left( \frac{B_{13}}{V_{13} / M_{13}} + \frac{T_{32}}{V_{32} / M_{32}} \right)$$

$$T_{32} = B_{32} \cdot \frac{v_{32}}{m_{32}} \cdot \left( \frac{T_{34}}{V_{34} / M_{34}} + \frac{B_{42}}{V_{42} / M_{42}} \right)$$

$$T_{34} = B_{34} \cdot \frac{v_{34}}{m_{34}} \cdot \left( \frac{B_{35}}{V_{35} / M_{35}} + \frac{B_{54}}{V_{54} / M_{54}} \right)$$

Assume traffics, circuits, congestion and overflow traffics as shown in the following table :

| Route | M | V | V/M | N | B | m | v | v/m |
|---|---|---|---|---|---|---|---|---|
| 1 - 2 | 30. | 30. | 1. | 30 | 0.132 | 3.97 | 12.15 | 3.06 |
| 1 - 3 | 100. | 150. | 1.5 | | 0.01 | - | - | - |
| 3 - 2 | 100. | 160. | 1.6 | 100 | 0.094 | 9.39 | 60.1 | 6.4 |
| 3 - 4 | 200. | 300. | 1.5 | 210 | 0.039 | 7.7 | 69.6 | 9.04 |
| 3 - 5 | 200. | 400. | 2. | | 0.01 | - | - | - |
| 5 - 4 | 300. | 500. | 1.67 | | 0.01 | - | - | - |
| 4 - 2 | 100. | 200. | 2. | | 0.01 | - | - | - |

The overall grade of service for traffic case 1 - 2 can then be calculated :

$$T_{34} = 0.039 \cdot 9.04 \cdot ( 0.01 / 2. + 0.01 / 1.67 ) = 0.0039$$

$$T_{32} = 0.094 \cdot 6.4 \cdot ( 0.0039 / 1.5 + 0.01 / 2. ) = 0.0046$$

$$T_{12} = 0.132 \cdot 3.06 \cdot ( 0.01 / 1.5 + 0.0046 / 1.6 ) = 0.0038$$

Similarly, the traffic case 1 - 5, for which there are no alternative routes, would experience a overall grade of service of

$$T_{15} = B_{13} \cdot \frac{1}{\frac{V_{13}}{M_{13}}} + B_{35} \cdot \frac{1}{\frac{V_{35}}{M_{35}}} = \frac{0.01}{1.5} + \frac{0.01}{2.} = 0.017$$

As we now know how to calculate the overall grade of service experienced by any traffic case from v/m, and V/M and B for the routes on the overflow path, we can set the proper values for these B so that the overall grade of service for all traffic cases is within the limits specified.

In the example above, consider that the required overall grade of service for traffic case 1 - 2, $GOS_{12} = 0.01$. That means that

$$T_{12} \leq GOS_{12}$$

or

$$\left( \frac{B_{13}}{\frac{V_{13}}{M_{13}}} + \frac{T_{32}}{\frac{V_{32}}{M_{32}}} \right) \leq \frac{GOS_{12}}{B_{12} \cdot \frac{v_{12}}{m_{12}}}$$

or, with the values used before,

$$\left( \frac{B_{13}}{1.5} + \frac{T_{32}}{1.6} \right) \leq \frac{0.01}{0.132 \cdot 3.06}$$

Assuming the 2 terms within the brackets to be of the same size, we get

$$B_{13} = \frac{1.5 \cdot 0.01}{2 \cdot 0.132 \cdot 3.06} = 0.019$$

and

$$T_{32} = \frac{1.6 \cdot 0.01}{2 \cdot 0.132 \cdot 3.06} = 0.020$$

In a similar way, the values for the other final routes are found to be

$B_{13}$ = 0.019
$B_{35}$ = 0.071
$B_{54}$ = 0.059
$B_{42}$ = 0.033

However, there are also other traffic cases to be considered, and, as stated before, the "worst" case should determine the GOS on the final routes.

If we then assume the required $GOS_{15}$ = 0.02, and make similar calculations, we find that $GOS_{13}$ and $GOS_{35}$ should be reduced to

$B_{13}$ = 0.015
$B_{35}$ = 0.020

The same exercise should then be carried out for any other traffic case using any of the final routes in its overflow path, and the relevant B-values will be set to satisfy all the GOS demands.

### 2.5.6 Non-Coincident Busy Hours

The various traffic flows in a network will rarely have their Busy Hour at exactly the same time. This is especially so for international networks, where the time difference between countries becomes an important factor in determining the traffic profiles. Similarly, in metropolitan networks the traffic for business subscribers will have its peak at a different time than residential traffic. Considerable savings in terms of circuits can be achieved if these traffic profiles are taken into account in dimensioning the inter-exchange network.

The dimensioning and optimization methods presented in the preceding sections of this chapter are valid or a certain point of time, ie the busy hour.

For networks with non-coincident busy hours ( NCBH ), the dimensioning and optimization procedure will be somewhat more complicated as we now have to work with ‘traffic profiles‘ instead of simple traffic values, thus introducing a new dimension, ie time, into our traffic model.

The notations given in the beginning of Chapter 2.5 will, except for N and k, be vectors, eg

A(t), M(t),...,B(t), t=1,2,...T.

The assumption is that during a given time period, t, we can work using the previously described methods of congestion calculation. This means that all formulae given in 2.5.1 and 2.5.2 can be used; we only have to affix the subscript t to the variables A, M, V, m, v, and B.

**Dimensioning**

A route offered traffic characterized by

$$\left.\begin{matrix} M(t) \\ V(t) \end{matrix}\right\} \; t = 1,2,\ldots\ldots\ldots\ldots T$$

can then be dimensioned for a specified grade of service, B(t), by finding the traffic yielding the highest number of circuits. It is obviously not necessary to investigate all points of time in detail, but only those with large traffics.

**Optimization :**

Considering anew the triangular routing pattern of Ch. 2.5.4, and adding the time parameter to the traffics involved, the optimisation equation then becomes

$$C(N) = c \cdot N + c_1 \cdot \Delta N_1 + c_2 \cdot \Delta N_2$$

where

$$\Delta N_1 = \frac{\partial N_1}{\partial M_1} \cdot \left[ \max_t \left\{ M_{10}(t) + m(N,t) \right\} - \max_t \left\{ M_{10}(t) \right\} \right]$$

$$\Delta N_2 = \frac{\partial N_2}{\partial M_2} \cdot \left[ \max_t \left\{ M_{20}(t) + m(N,t) \right\} - \max_t \left\{ M_{20}(t) \right\} \right]$$

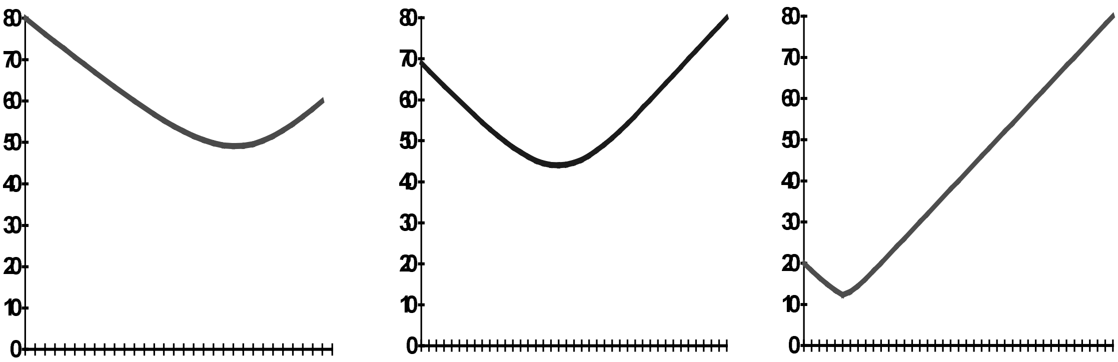
and $M_{10}(t)$ = traffic mean offered to route i-T at time t, not including overflow traffic from i-j
$M_{20}(t)$ = ditto for route T-j

The derivative $\frac{\partial N_1}{\partial M1}$ is calculated in the case of

- low-loss routes : with constant congestion value, and for the point of time, t, which determined $N_1$ ;
- high-usage routes : with constant overflow traffic mean at the point of time, t, with the largest value.

As in the case of coincident busy hours, the optimal value of N is found by comparing the values of **C**(N) for $N = 0,1,2,\ldots\ldots,N_D$.

As an illustration of the difference in the cost, and circuit number, using the NCBH concept, compare direct route, i-j, and the tandem routes, i-T and T-j. In the other two cases, the traffic profile on the tandem routes are the same, but the traffic peak on the direct route coincides with high traffics on the tandem routes in the figure at center, and coincides with low traffics on the tandem routes in the figure at the right.

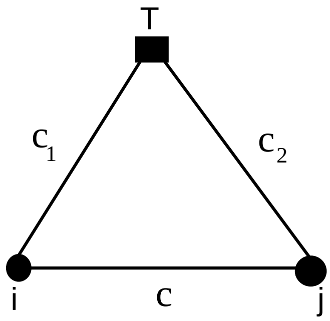


**<u>Assumtions :</u>**

c=1

$$c_1 \cdot \frac{\partial N_1}{\partial M_1} = 0.9$$

$$c_2 \cdot \frac{\partial N_2}{\partial M_2} = 1.1$$




| | Traffics in erlang at time t | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| t = | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 |
| i - j | 40 | 20 | 30 | 40 | 5 | 5 | 10 | 5 | 5 | 10 | 20 | 30 | 40 |
| i - T | 100 | 50 | 80 | 100 | 90 | 80 | 60 | 50 | 80 | 100 | 90 | 80 | 60 |
| T - j | 100 | 50 | 100 | 90 | 80 | 75 | 70 | 50 | 100 | 90 | 80 | 75 | 70 |
| Result of optimi-sation : | **Nij=42 C(N)=49** | **Nij= 36 C(N)=44** | | | | | | **Nij=10 C(N)=12.3** | | | | | |

### 3.6 CALCULATION OF DERIVATIVE, $\frac{\partial N}{\partial M}$

When dimensioning or optimising the number of circuits in a route, it is also necessary to evaluate the *efficiency* of the route, ie the partial derivative of the number of circuits with regard to the mean of the offered traffic, $\frac{\partial N}{\partial M}$. In the next iteration of the planning programs these derivatives will then be used for various purposes,ie

- optimization of exchange area boundaries;
- optimization of routes on lower levels of the network hierarchy;
- evaluating changes in the junction network cost when introducing new exchanges;
- optimising availabilities of individual routes from exchanges with restricted availability.

For the general case of non-Poisson traffic, characterized by the mean, M, and the variance, V, and restricted availability, described by a system dependent blocking function, $W_i$, the route congestion as a function of these parameters can be written as

B(M,V,N,W)

where N is the number of circuits. It will be more convenient, however, to use ( M , V/M ) instead of ( M , V ) to describe the traffic offered; in the formulae in this chapter this ratio will be denoted by Θ. For the calculation of derivatives, Θ can be assumed to be approximately constant. The use of Θ instead of V facilitates the necessary calculations, without affecting the accuracy of the results significantly. Thus, the congestion on the route will be denoted

B(M, Θ,N,W)

Two cases have to be considered, routes with overflow possibility, and routes without.

For the case of direct, *low-loss routes* without overflow possibility, the route congestion has a certain pre-defined value, in other words,

B(M, Θ,N,W) = constant

Partial derivation of B with regard to M yields

$$\frac{\partial B}{\partial M} + \frac{\partial B}{\partial N} \cdot \frac{\partial N}{\partial M} = 0$$

or

$$\cdot \frac{\partial N}{\partial M} = - \frac{\frac{\partial B}{\partial M}}{\frac{\partial B}{\partial N}}$$

For the case of *high-usage routes*, ie routes with overflow possibility, the overflow traffic from the route should be considered constant rather than the congestion, so that

m = M * B(M, Θ, N, W) = constant

In this case partial derivation with regard to M yields

$$B + M \cdot \left( \frac{\partial B}{\partial M} + \frac{\partial B}{\partial N} \cdot \frac{\partial N}{\partial M} \right) = 0$$

or

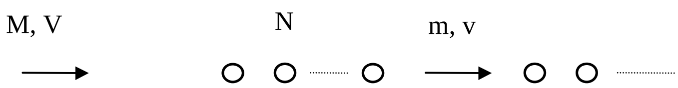
$$\cdot\frac{\partial N}{\partial M} = -\frac{\frac{B}{M} + \frac{\partial B}{\partial M}}{\frac{\partial B}{\partial N}}$$

We then have to find suitable expressions for $\frac{\partial B}{\partial M}$ and $\frac{\partial B}{\partial N}$ for the cases of

- full availability (Θ =1 and Θ > 1 )
- restricted availability, Poisson traffic (Θ = 1 )
- restricted availability, degenerated traffic (Θ > 1 )

*3.6.1 Full availability*

We now want to find the partial derivatives $\frac{\partial B}{\partial M}$ and $\frac{\partial B}{\partial N}$ for the configuration shown below :




As in Chapter 2.5.1, **Congestion Theory, Full availability,Alternative Routes**, this configuration is replaced by


The parameters N* and A* of this "equivalent group" can be determined from the equation system

$$M = A^* \cdot E_{N^*}(A^*)$$

$$\Theta = 1 - M + \frac{A^*}{N^* + 1 - A^* + M} \qquad (3.6.1)$$

The mean of the overflow traffic from the alternative route is then estimated as

$$m = A^* \cdot E_{N+N^*}(A^*)$$

The congestion, B, can then be written as

$$B(M, \Theta, N) = \frac{m}{M} = \frac{E_{N+N^*}(A^*)}{E_{N^*}(A^*)} \qquad (3.6.2)$$

and therefore,

$$\frac{\partial B}{\partial M} = \frac{\partial B}{\partial N^*} \cdot \frac{\partial N^*}{\partial M} + \frac{\partial B}{\partial A^*}\frac{\partial A^*}{\partial M}$$

Partial derivation of (3.6.1) with regard to M gives

$$\frac{\partial A^*}{\partial M} \cdot \frac{M}{M+\Theta-1} + \frac{\partial N^*}{\partial M} \cdot \frac{\partial E_{N^*}(A^*)}{\partial N^*} \cdot A^* = 1$$

$$\frac{\partial A^*}{\partial M} \cdot \frac{M+\Theta}{M+\Theta-1} - \frac{\partial N^*}{\partial M} = 1 + \cdot \frac{A^*}{(M+\Theta-1)^2}$$

from which $\frac{\partial A^*}{\partial M}$ and $\frac{\partial N^*}{\partial M}$ can be calculated.

From (3.6.2) we obtain

$$\frac{\partial B}{\partial A^*} = B \cdot \left( \frac{N}{A^*} - E_{N^*}(A^*) + E_{N+N^*}(A^*) \right)$$

$$\frac{\partial B}{\partial N^*} = \frac{1}{E_{N^*}(A^*)} \cdot \left( \frac{\partial E_{N+N^*}(A^*)}{\partial (N^*+N)} - B \cdot \frac{\partial E_{N^*}(A^*)}{\partial N^*} \right)$$

$$\frac{\partial B}{\partial N} = \frac{1}{E_{N^*}(A^*)} \cdot \frac{\partial E_{N+N^*}(A^*)}{\partial (N^*+N)}$$

The remaining derivatives $\frac{\partial E_{N^*}(A^*)}{\partial N^*}$ and $\frac{\partial E_{N+N^*}(A^*)}{\partial (N^*+N)}$ are treated in Chapter 3.9.2, **Erlang's formula for non-integer number of circuits.**

### *3.6.2 Restricted availability, Poisson traffic*

As discussed in chapter 2.5.2, the congestion B(N,M) can be written as

$$B(N,M) = \sum_{i=0}^{N} W_i \cdot P(i)$$

where P(i) can be determined from the relations

$$P(i) \cdot M \cdot (1-W_i) = P(i+1) \cdot (i+1)$$

$$\sum_{i=0}^{N} P(i) = 1 \qquad \text{(i=0,1,2,...N-1)}$$

Derivation yields

$$\frac{\partial B}{\partial M} = \frac{1}{M} \cdot \sum_{k=0}^{N} k \cdot W_k \cdot P(k) - B \cdot (1-B)$$

For $\frac{\partial B}{\partial N}$ no explicit formula is available; a suitable approximation is

$$\frac{\partial B}{\partial N} \approx -\frac{(B(N-1) - B(N+1))}{2}$$

### *3.6.3 Restricted availability,non-Poisson traffic*

Explicit formulae are not available for this case. An approximation similar to the one used in chapter 2.5.2 for overflow traffics and circuits has been found to be adequate for our needs :

$$\left(\frac{\partial N}{\partial M}\right) \approx \left(\frac{\partial N}{\partial M}\right)_2 + \left(\frac{\partial N}{\partial M}\right)_3 - \left(\frac{\partial N}{\partial M}\right)_1$$

where the subscripts indicate the simpler cases, that is

1 : Full availability , $\Theta = 1$

2 : Full availability , $\Theta > 1$

3 : Restricted availability , $\Theta = 1$

## 3.7 CALCULATION OF WILKINSON'S EQUIVALENT GROUP

In some of the chapters dealing with dimensioning and /or optimisation of circuits, use has been made of Wilkinson's Equivalent Group parameters, $N^*$ and $A^*$. These parameters can be found by solving the following system :

$$M = A^* \cdot E_{N^*}(A^*)$$

$$V = M \cdot \left(1 - M + \frac{A^*}{N^* + 1 - A^* + M}\right)$$

for given values of M and V/M ( M > 0 , V/M > 1 ). This can be done in the following way :

$N^*$ is expressed as a function of M, V/M and $A^*$, ie

$$N^* = \frac{A^*}{V / M + M - 1} + A^* - M - 1$$

thus reducing the problem to one independent variable, $A^*$.

We can then use Newton-Raphson's method to solve the remaining equation,

$$M = A^* \cdot E_{N^*}(A^*)$$

or

$$f(A^*) = M - A^* \cdot E_{N^*}(A^*) = 0$$

in the usual way, that is by finding a suitable starting value for A]*], A[o[, and improving it iteratively by using

$$A_{K+1} = A_K - \frac{f(A_K)}{f'(A_K)}$$

until the resulting values become close enough to M and V/M.

A suitable starting value for $A^*$ has been given by Y. Rapp as

$$A_0 = V + 3 \cdot V / M \cdot (V / M - 1)$$

The calculation of $f'(A)$ remains, and we get

$$-f'(A) = E_N(A) + A \cdot \left(\frac{\partial E_N(A)}{\partial A} + \frac{\partial E_N(A)}{\partial N} \cdot \frac{\partial N}{\partial A}\right)$$

or

$$-f'(A) = E_N(A) \cdot \left(1 + N - A + A \cdot E_N(A)\right) + A \cdot \frac{\partial E_N(A)}{\partial N} \cdot \frac{V / M + M}{V / M + M - 1}$$

For calculation of $\frac{\partial E_N(A)}{\partial N}$ see Chapter 3.9.2 '**Erlang's formula for non-integer number of circuits**'.

<u>Practical hint</u> : to avoid numerical problems which can occur for some combinations of M and V/M, it is advisable to ensure convergence of this method by saving the current lower and upper limits for A, checking A[K+1[ against these limits, and, if necessary, use halving of this interval to correct A should it fall outside the interval.

## 3.8 BLOCKING FUNCTIONS

In a restricted availability system the connection paths are so arranged that a call may be unsuccessful even when there are still idle circuits in the called group. Thus the arrangements known as gradings and link systems, as well as combinations of these two will generally be classified as restricted availability systems.

Calls may overflow from the primary group even when it has free circuits. The overflows are governed by a *loss function,* $W_i$, which is defined as the probability that a call arriving, when there are $\boldsymbol{i}$ occupied primary circuits, will be rejected.

The *loss function* for a given system depends clearly on the way the connection paths have been arranged, and is therefore much dependent on the type of switching equipment used. Thus it is necessary to construct these loss functions for any type of switching equipment to be encountered in the planning of a specific network, and to program the necessary routines.

Below is an example of such a loss function for Erlang's Ideal Grading.

### *Erlang's Ideal Grading*

Using the assumption that the proceding calls are distributed at random among the outgoing circuits, Erlang arrived at his well-known formula

$$W_i = \frac{\binom{i}{K}}{\binom{N}{K}}$$

where N is the total number of outgoing circuits, and K is the number of circuits available from each inlet group.

The calculation of the W-values should be done according to the following recursion formula to avoid numerical problems:

$$W_i = 1$$

$$W_{i-1} = W_i \cdot \frac{i-K}{i} \qquad \text{for I = N, N-1, N-2, ........,K}$$

### *Other arrangements*

For other arrangements of the connection paths, especially for two or three stage link systems, the formulae for $W_i$ are much more complicated. For some arrangements, calculation schemes have been developed and programmed. Wallström's "Congestion Studies in Telephone Systems with Overflow Facilities" describes several such functions.

## 3.9 ERLANG'S FORMULA

In the description of methods for optimising and dimensioning of the inter-exchange network there are numerous references to Erlang's formula, both for integer and non-integer number of circuits, and the derivatives thereof with regard to circuits and/or offered traffic. This chapter deals with the numerical problems encountered in this context, and describes ways arriving at numerical values of these entities for given parameters.

### *3.9.1 Erlang's formula for integer number of circuits*

For an integer number of circuits, N, and offered traffic mean, A, Erlang's formula is usually written as

$$E_N(A) = \frac{\frac{A^N}{N!}}{\sum_{i=0}^{N} \frac{A^i}{i!}}$$

This notation is, however, quite unsuitable for use in computer programs especially for higher values of N. Separate calculations for $\frac{A^N}{N!}$ and the sum of such terms will result in such large values that the result is numerically useless. It can be shown quite easily that E[N[(A) can be recursively calculated from the following formula :

$$E_K(A) = \frac{A \cdot E_{K-1}(A)}{K + A \cdot E_{K-1}(A)}$$

with

$$E_0(A) = 1$$

The partial derivative $E_N(A)$ of with regard to A is found to be

$$\frac{\partial E_N(A)}{\partial A} = E_N(A) \cdot \left( \frac{N}{A} - 1 + E_N(A) \right)$$

The partial derivative of $E_N(A)$ with regard to N has no mathematical meaning as long as we consider $E_N(A)$ as defined only for integer values of N. In this case we can approximate the derivative with a difference,

$$\frac{\partial E_N(A)}{\partial N} \approx E_{N+1}(A) - E_N(A)$$

or

$$\frac{\partial E_N(A)}{\partial N} \approx \frac{E_{N+1}(A) - E_{N-1}(A)}{2}$$

or other expressions using higher order differences.

There is, however, a definition of $E_N(A)$ for real N, that is

$$E_N(A) = \frac{A^N \cdot e^{-A}}{\int_A^{\infty} t^N e^{-t}\, dt}$$

Derivation of this expression with regard to N yields

$$\frac{\partial E_N(A)}{\partial N} = -E_N(A)\cdot\Psi_{N+1}(A)$$

where

$$\Psi_1(A) = \int_A^\infty e^{A-t}t^{-1}dt$$

and $\Psi_{N+1}(A)$ can be found recursively from

$$\Psi_{N+1}(A) = \left(1 - E_N(A)\right)\cdot\left(\Psi_N(A) + \frac{1}{K}\right) \qquad \text{(K = 1,2,....,N)}$$

Numerical calculation of $\Psi_1(A)$ can be carried out according to one of the two following approximations:

*Approximation I :*

For *small* values of A ( 0 < A < 5 ) :

$$\Psi_1(A) = e^{-A}\cdot\left(C + \log(A) + \sum_{K=1}^{\infty}\frac{-A^K}{K\cdot K!}\right)$$

where C = 0.57721566490... is Euler's constant.

For *large* values of A ( 15 < A ) :

$$\Psi_1(A) = -\sum_{K=1}^{n}\frac{(-1)^K\cdot(k-1)!}{A^K} + R_n$$

where $R_n < \dfrac{n!}{A^{n+1}}$

*Approximation II*, yielding 8 correct decimals :

For A < 1 :

$$\Psi_1(A) = e^{-A}\cdot\left(\left(\left(\left(\left(a_1\cdot A - a_2\right)\cdot A + a_3\right)\cdot A - a_4\right)\cdot A + a_5\right)\cdot A - a_6 - \log(A)\right)$$

For A > 1 :

$$\Psi_1(A) = \frac{\left(\left(\left(A + b_1\right)\cdot A + b_2\right)\cdot A + b_3\right)\cdot A + b_4}{\left(\left(\left(\left(A + c_1\right)\cdot A + c_2\right)\cdot A + c_3\right)\cdot A + c_4\right)\cdot A}$$

where

| | | |
|---|---|---|
| $a_1$ = 0.00107857 | $b_1$ = 8.5733287401 | $c_1$ = 9.5733223454 |
| $a_2$ = 0.00976004 | $b_2$ = 18.059016973 | $c_2$ = 25.6329561486 |
| $a_3$ = 0.05519968 | $b_3$ = 8.6347608925 | $c_3$ = 21.0996530827 |
| $a_4$ = 0.24991055 | $b_4$ = 0.2677737343 | $c_4$ = 3.9584969228 |
| $a_5$ = 0.99999193 | | |
| $a_6$ = 0.57721566 | | |

*3.9.2 Erlang's formula for non-integer number of circuits*

The definition of $E_{N+x}(A)$ ,where N is a non-negative integer and 0 < x < 1, is given by

$$E_{N+x}(A) = \frac{A^{N+x} \cdot e^{-A}}{\int_A^\infty t^{N+x} \cdot e^{-t}\, dt}$$

A recursion formula is readily found, yielding

$$E_{K+x}(A) = \frac{A \cdot E_{K+x-1}(A)}{K + x + A \cdot E_{K+x-1}(A)}$$

with

$$E_x(A) = \frac{A^x \cdot e^{-A}}{\int_A^\infty t^x \cdot e^{-t}\, dt}$$

For A < 1 , the integral can be re-written as

$$\int_A^\infty t^x e^{-t}\, dt = \Gamma(x+1) - \sum_{K=0}^{\infty} \frac{(-1)^K \cdot A^{x+K+1}}{K! \cdot (x+1+K)}$$

where

$$\Gamma(x+1) = \prod_{K=1}^{\infty} \frac{\left(1 + \frac{1}{K}\right)^x}{1 + \frac{x}{K}}$$

For A > 1 , the convergence of the sum may not be all that good, and the following algorithm, *called continued fractions expansion*, should be used to find $E_x(A)$ :

Step 1 : Set C = 1
Set D = 1
Set R = 1
Set I = 0

Step 2 : Set B = I/2 - x for even I
= (I+1)/2 for odd I

Set D = A/(A+B*D)

Set R = R*(D-1)

Set C = C + R

Step 3 : If $C \cdot |R| < \varepsilon$ then $E_x(A) = \frac{1}{C}$

$C \cdot |R| \geq \varepsilon$ then set I = I+1 ,and continue from Step 2

The partial derivative of $E_{\ \ +x}(A)$ with regard to A is, as before,

$$\frac{\partial E_{N+x}(A)}{\partial A} = E_{N+x}(A) \cdot \left( \frac{N+x}{A} - 1 + E_{N+x}(A) \right)$$

The partial derivative of $E_{\ \ +x}(A)$ with regard to (N+x) is

$$\frac{\partial E_{N+x}(A)}{\partial (N+x)} = -E_{N+x}(A) \cdot \Psi_{N+x+1}(A))$$

where $\Psi_{N+x+1}(A)$ can be recursively calculated from

$$\Psi_{K+x+1}(A) = \left(1 - E_{K+x}(A)\right) \cdot \left( \Psi_{K+x}(A) + \frac{1}{K+x} \right) \qquad \text{K = 1,2,........,N}$$

and, finally, $\Psi_{x+1}(A)$ can be obtained approximately from

$$\Psi_{x+1}(A) \approx \frac{E_{x-\Delta}(A) - E_{x+\Delta}(A)}{2 \cdot \Delta \cdot E_x(A)}$$

with Δ a small value ( 0 < Δ < x ).