Mathematics of Money

and

Economy Study Techniques

by Mr. G.Moumoulidis (OTE)

UNION INTERNATIONALE DES TELECOMMUNICATIONS
INTERNATIONAL TELECOMMUNICATION UNION
UNION INTERNACIONAL DE TELECOMUNICACIONES

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Contents

1. Mathematics of Money

1.1 *Introduction*

Money can be used to earn more money. Money can be placed in a bank savings account, where it will earn interest. The mathematics of money is based on the fact that all money works and, therefore, has potential earning power.

This earning power of money can also be viewed as a cost of using money. The term *interest* is commonly used to denote the rate of money's earning power.

Money has earning power because it works over a period of time. Before the actual return from the investment of money can be realised, time must pass. The entire concept of the earning power of money can therefore be viewed as the time value of money. An analyst must be able to apply this concept when estimating the cost of alternative ways to accomplish a single business goal. Often, the estimated expenditures for alternative plans will occur in varying amounts and at varying times.

1.2 *Money and time*

If an amount A is placed in a bank paying an interest rate i per year, it grows to $A(1+i)$ at the end of one year. So, today, the amount A at i interest is equivalent to $A(1+i)$ a year from now. The initial deposit and the earned interest left have earning power because interest is *compounded*. This means that it is computed on the capital plus the accrued interest. When the second year begins, the amount in the bank is $A(1+i)$, and it will earn $iA(1+i)$ at the end of the second year. The original amount A will then have a value $A(1+i)^2$. Consequently, based on the above reasoning, the cost of using money is intimately associated with time and cannot be considered independently of it.

1.3 *Cash flow diagrams*

Cash flow diagram is simply a horizontal line marked off to represent periods of time. Upward arrows usually represent the receipts of cash and downward arrows represent expenditures outflows. Borrowing money from the bank and repaying it in four installments would therefore be shown in Figure 1.

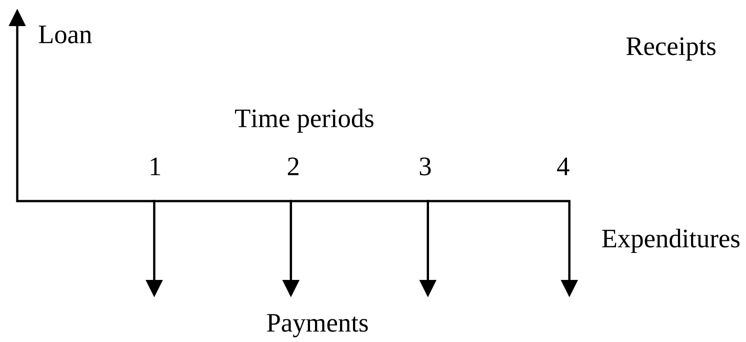


Figure 1

1.4 *Derivation and applications of time-value factors*

1.4.1 *The future worth of a present amount (F/P)*

The future worth of a present amount is the sum at the end of a specific period of time that this present amount of money will accumulate to at a given rate of interest each compounding period. The future worth of a present amount P may be evaluated as follows:

Future worth at the end of the first year

$$(F)_1 = p + p \cdot i = p(1+i)$$

Future worth at the end of the second year

$$(F)_2 = p(1+i) + ip \cdot (1+i) = p(1+i)^2$$

Future worth at the end of *N*th year

$$(F)_N = P(1+i)^N$$

The time value factor (*F/P)*, the ratio of the future amount to the present amount is then expressed as follows:

$$\left(F / P\right)_N^i = (1+i)^N \qquad \textbf{1}$$

Example 1:

Find the future worth of 100 monetary units (MU) at the end of ten years from now, at 10 % interest rate.

Solution:

We have:

$$F = P \cdot (F / P)_{10}^{10\%} = P(1+0.1)^{10}$$
$$F = 100(1.1)^{10} = 259.4\,MU$$

1.4.2 *The present worth of a future amount (P/F)*

The present worth of a future amount is the amount of money at the start of a specific period of time, at a given rate of interest. In the previous section, we found that:

$$F = P(1+i)^N$$

which becomes

$$P = F / (1+i)^N = P(1+i)^{-N}$$

The *time-value factor* for the present worth of a future amount is therefore expressed as follows:

$$(P / F)_N^{i\%} = 1 / (1+i)^N \qquad \textbf{2}$$

1.4.3 *The future worth of an annuity (F/A)*

An annuity *A* is simply a series of equal periodic payments for a specified number of periods. For payments occurring at the beginning of each period, we have the following diagram (Figure 2):

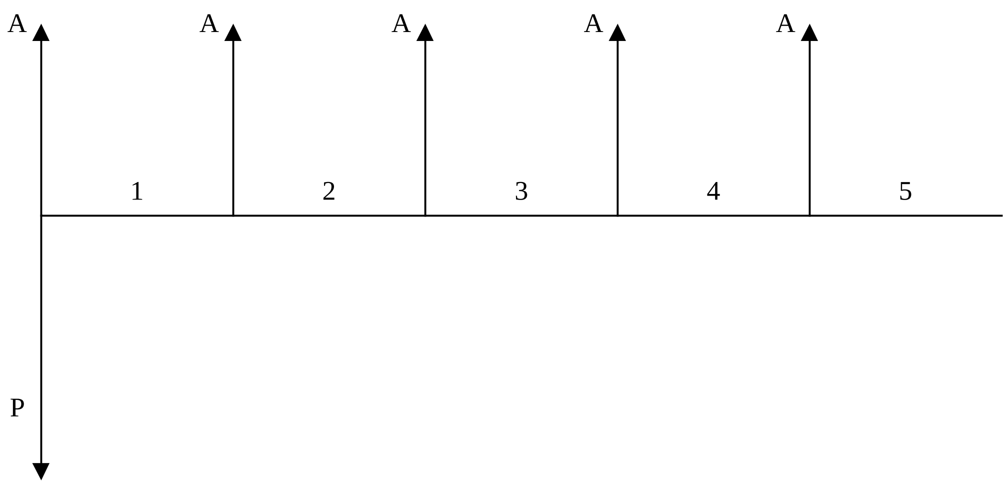


Figure 2

The upward A arrows represent equal payments and the downward P arrow represents the present worth of the annuity (the series of equal periodic payments).

One case often used is when payments take place at the end of the period. In Figure 3, a diagram for an end of period annuity is shown:

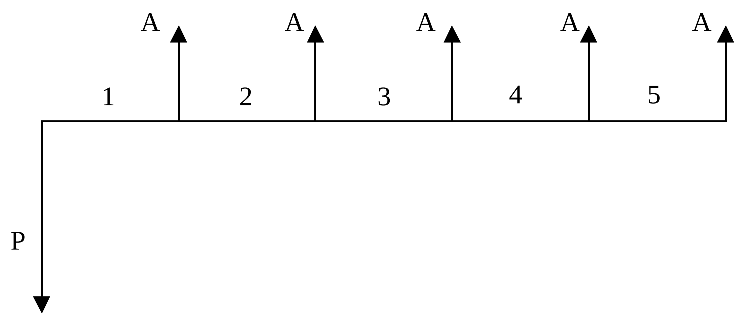


Figure 3

The present worth P of an end-of-period over N years is given by:

$$P = A \cdot \frac{(1+i)^N - 1}{i(1+i)^N} \qquad \mathbf{3}$$

The time-value factor for the present worth of an annuity is:

$$\left(P / A\right)_N^{i\%} = \frac{(1+i)^N - 1}{i(1+i)^N} \qquad \mathbf{3\ a}$$

<u>*Example 2*</u>:

Find the present worth of a 100 MU annuity at 8 % interest rate, at the end of 10 years.

<u>*Solution*</u>:

We have:

$$\left(P\ /\ A\right)_{10}^{8\%}=\frac{(1+0.08)^{10}-1}{0.08(1+0.08)^{10}}=\frac{2.159-1}{0.08\times 2.159}=6.71$$

The present worth is:

$$P=A\times 6.71=100\times 6.71=671\,MU$$

1.4.3.a Annuity *A,* from a present amount.

The annuity *A*, from a present amount, is the annual amount that can be withdrawn for a specified number of periods. If we solve (3) with respect to A, we have:

$$A=P\cdot\frac{i(1+i)}{(1+i)\quad -1} \qquad \textbf{4}$$

This gives the annuity from a present amount. The time value factor is

$$\left(A\ /\ P\right)_{N}^{i\%}=\frac{i(1+i)^{N}}{(1+i)^{N}-1} \qquad \textbf{4 a}$$

<u>*Example 3*</u>:

Determine the annuity over a ten-year period of time which is equivalent to a present worth of 671 *MU* at 8 % interest.

<u>*Solution*</u>:

The equation (4) is

$$A=671\cdot\frac{0.08(1+0.08)^{10}}{(1+0.08)^{10}-1}=671\times 0.149=100\,MU$$

1.4.4 *The future worth of an annuity (F/A)*

The future worth of an annuity for a given period of time is the sum at the end of the specified time of the future worth of all payments at a given rate of interest.

The general equation is:

$$F=A\cdot\frac{(1+i)^{N}-1}{i} \qquad \textbf{5}$$

The time-value factor for the future worth of an annuity is then expressed as:

$$\left(F\ /\ A\right)^{\%}=\frac{(1+i)\quad -1}{i} \qquad \textbf{5 a}$$

<u>*Example 4*</u>:

Find the future worth of a 100 *MU* annuity at 8 % interest at the end of 10 years.

<u>*Solution*</u>:

Using equation 5, we have:

$$F = A\frac{(1+i)^N - 1}{i} = 100 \cdot \frac{(1+0.08)^{10} - 1}{0.08} = 1448.6\ MU$$

The annuity from a future amount (*A/F*) is the amount *A* which, if it is set aside each year, will accumulate to a known sum at the end of a specific period of time, at a given rate of interest.

Solving the equation (5) with respect to *A*, we get:

$$A = F\frac{i}{(1+i)^N - 1} \qquad \textbf{6}$$

which provides the required annuity.

The time-value factor is then as follows:

$$\left(A/F\right)_N^{i\%} = \frac{i}{(1+i)^N - 1} \qquad \textbf{6 a}$$

1.5 *Effective and nominal rates of interest-continuous compounding*

So far, we have dealt with the derivation of formulae of periodic compounding with end-of-periods of payment. However, payment does not always occur annually and interest can be compounded at many different intervals.

Assume an amount *A* is compounded every 6 months at nominal interest rate *r* per year. We calculate the future worth of *A MU* at the end of the first year.

At the end of the first six-month period, we have

$$A(1+r/2)$$

and at the end of the second six-month period

$$A \cdot (1+r/2)^2$$

if the amount *A* is compounded quarterly, at the end of the first year we have:

$$A(1+r/4)^4$$

Eventually, if the compounding occurs *M* times a year, at the end of the year we get:

$$A(1+r/M)^M$$

The effective *i* rate per period is associated with the nominal through the relation:

$$(1+i) = (1+r/M)^M \qquad \textbf{7}$$

The relation (7) provides the conversion of rates.

Example 5:

What effective annual rate corresponds to a nominal rate of 8 % per annum compounded monthly?

Solution

We have here $r = 8\ \%$ and $M = 12$, and subsequently:

$$1+i=\left(1+\frac{0.08}{12}\right)^{12}=1.083 \Rightarrow$$
$$i=8.3\%$$

Example 6:

If the effective rate is 8 %, what is the nominal under monthly compounding?

Solution

If we have $i = 8\ \%$ and $M = 12$ solving (7) for r, we get:

$$(1+i)=(1+r\ /\ M)^{M} \Rightarrow$$
$$1+0.08=(1+r\ /\ 12)^{12} \Rightarrow$$
$$\sqrt[12]{1.08}=1+r\ /\ 12 \Rightarrow$$
$$r=12\left[\sqrt[12]{1.08}-1\right]=0.0772 \Rightarrow$$
$$r=7.72\%$$

If we let M increase to infinite, the relation (7) to the limit becomes:

$$1+i=e^{r} \qquad \textbf{7 a}$$

where r corresponds to nominal interest rate for *continuous compounding*, and is called *force of interest*.

1.6 Applications of continuous compounding

Continuous compounding is extensively used in engineering economy studies. It facilitates extremely the operation. Sums of discrete terms are then converted to integrals which are readily evaluated. The terms $(1+i)$ are transformed to e and i or r.

The actual cash flow generally in telecommunications administrations, both inwards from receipts and outwards for expenditures, are continuous.

The present worth of the annuity illustrated in Figure 4 is evaluated by adding up the present worth of each individual annual cost.

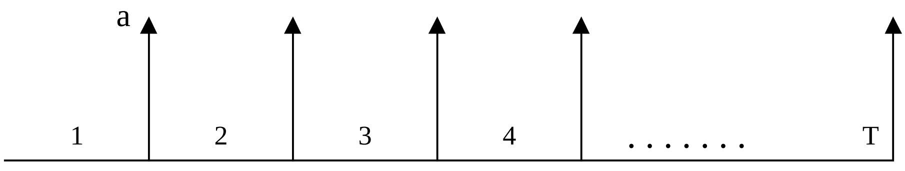


Figure 4

$$PW=\sum_{t=1}^{T}a(1+i)^{-t} \qquad \textbf{8}$$

In case of continuous compounding, the discrete costs are becoming continuous, as illustrated in Figure 5.

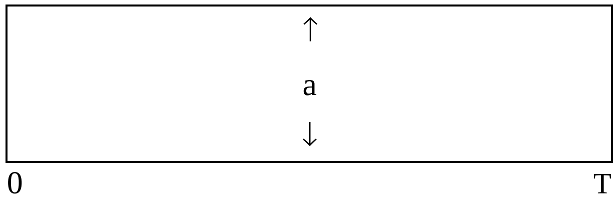


Figure 5

The present worth of the continuous annuity is given by:

$$PW = \int_{o}^{T} ae^{-rt}\,dt \qquad \mathbf{9}$$

The similarity between the relationships (8) and (9) is evident. If we interchange the sum by integral and $(1+i)^{-t}$ by e^{-rt}, the first formula is transformed into the other.

The ease with which an integral can be evaluated provides the great advantage of the formula (9) with respect to (8). When cash flow is a function of time $a = a(t)$, the integral (9) becomes:

$$PW = \int_{o}^{T} a(t)e^{-rt}\,dt \qquad \mathbf{10}$$

This case is often encountered in pair gain system (line concentrators, single channel carriers, subscriber *PCM*, etc.) applications in loop plants.

2. Economy study techniques

2.1 *General consideration*

The basic economic study methods are:

1) the present worth method
2) the annuity method
3) the rate of return method.

The choice of method to be employed for a certain study is rather arbitrary and the ease of calculation and the simplicity of presentation are factors which should always be considered when making the decision. In network planning optimization problems, the most attractive method is the present worth technique.

2.2 *Present worth (PW) method*

The preponderant method in optimization problems is the present worth method. This method refers to all the economy events, both incomes and expenditures, to the same time point and expresses them in the form of one figure. When comparing different alternatives which provide the same revenues or cost savings, it is only sufficient to select the alternative which has the least present worth of all expenditures or annual charges. There are two methods of present worth:

- the present worth of expenditures (PWE) and
- the present worth of annual cost (PWAC).

2.3 *Present worth of expenditures (PWE)*

The present worth of expenditures (PWE) method is a measure of how attractive an alternative is from the point of view and of how much money one administration must spend to undertake each alternative.

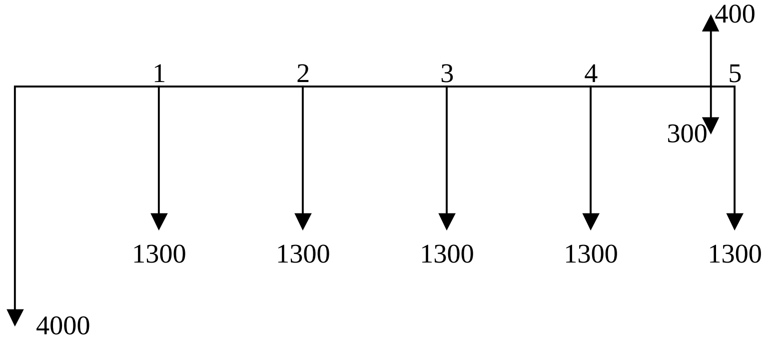
By finding the PWE of each alternative, we select the one with the least PWE, provided that each alternative provides the same service to subscribers. The PWE method does not require any estimate of revenues; however, if a difference in revenues is anticipated because of a difference in service, then these anticipated revenues must be considered in order to maintain comparability.

Example 7

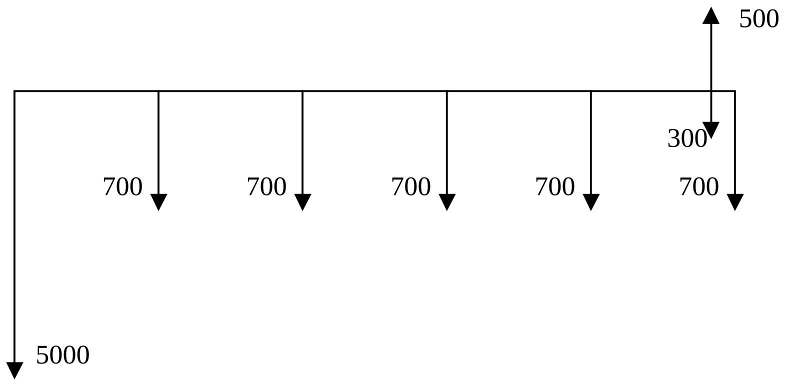
Assume two alternatives for which we have the following table:

Alternative	A	B
First cost	4000 mu	5000 mu
Gross salvage	400	500
Cost of removal	300	300
Operation and maintenance cost/year	1300	700
Service life	5 years	5 years
Interest rate	10 %	

We want to find which alternative is the most attractive as far as expenditures are concerned. It is a good practice to organize the cash flow diagram for each alternative. In Figure 5 (a, b), cash flows are illustrated.



Alternative A
Figure 5-a



Alternative B
Figure 5-b

The salvage arrow is pointing upwards since it is a receipt.

PWE calculations:

Alternative A:

- PWE of operating and maintenance =

$$1300 \cdot \frac{(1+0.1)^5 - 1}{0.1(1+0.1)^5} = 4928\,MU$$

- PWE of net salvage =

$$-(400-300)(P/F)_5^{10\%} = -100/(1+0.1)^5 = -62\,MU$$

PWE of first cost = *4000 MU*

Total PWE = *4000 + 4928 - 62 = 8866 MU*

Alternative B:

- PWE of operating and maintenance =

$$700 \times (P/A)_5^{10\%} = 700\frac{(1+i)^N - 1}{i(1+i)^N} = 700\frac{(1+0.1)^5 - 1}{0.1(1+0.1)^5} = 2654\,MU$$

- PWE of net salvage =

$$-(500-300)(P/F)_{5}^{10\%}=-124$$

- PWE of first cost = *5000 MU*

Total PWE = *5000 + 2654 - 124 = 7530 MU*

Comparing the PWE of each alternative, we choose the alternative *B* since it has the least PWE.

2.4 *Present worth of annual cost (PWAC)*

The present worth of annual costs (PWAC) method is essentially the same as the PWE method, except that capital costs are converted to equivalent annual costs (AC) before their worth is found. Operations costs should be treated as they are expected to occur. In many instances, results of a PWAC analysis will be exactly the same as those for a PWE analysis because the present worth of an annuity, which is equivalent to an expenditure, is the expenditure itself. Therefore, if the present worth of the annual costs is found over a period equal to the life of the plant, the PWE equals the PWAC. Of any number of mutually exclusive alternatives for doing a specific job, pick as the most economic alternative the one with the lowest PWAC at the cost of money.

PWAC is very popular and the reason is that working with annual costs can simplify the treatment of non-coincident equipment placements and retirements.

Many studies are not coterminated. That is, some plant is expected to live longer in one plan than in another. The PWAC method is most advantageous in such studies.

2.5 *The annuity method*

With this method, initial capital costs are converted to equivalent annual costs. Constant annual receipts and/or operating costs are then substracted and/or added to the annual capital costs. Residual value is assumed to be zero.

The application of the annuity method is limited by the assumptions and conditions:

- All investments have to be made at one time, at the beginning of the calculation period;
- Operating expenses and receipts have to remain constant during the calculation period;
- Residual value is zero.

These difficulties can be avoided by using the present worth method. For each plant item, total annual costs are determined and then converted to present values.

Example 8

Consider a route which is experiencing growth and whose existing capacity is exhausted. Any of the following alternatives may be appropriate:

- Place a new cable.
- Place one or more pair gain systems as permanent solution, using existing pairs as links.
- Place one or more pair gain systems, temporarily, using existing cable pairs as links. When these systems are exhausted, remove them and place a new cable.

The first alternative is the classic “all cable” solution. The second alternative is often called “a permanent pair gain” solution since the pair gain solution is not removed. The last alternative is called “a temporary pair gain” solution in which the relief cable is deferred, but once it is placed the pair gain systems are removed. A complete study of this problem is rather tricky. We will only restrict ourselves to the third alternative in its simplified form.

The total annual charge to place a new cable to meet the demand in a certain route is *3500 MU*. The total annual charge to install a pair gain system is *1200 MU/year*, but every year we have to install one more pair gain system in order to fulfil the demand. The interest rate is taken equal to 10 %.

The cash flow for each case is illustrated in Figure 6 (a & b). The size of arrows in case (a) (buried cable) is constant, which means that the whole investment was realized at zero time. Unlike to the buried cable, the annual charges for the pair gain alternative increase linearly because of the fact that every year one more system is then to be installed to meet demand growth (see Figure 6).

Looking as the cash flow, we easily conclude that an alternative is advantageous as long as annual charges remain less than the others.

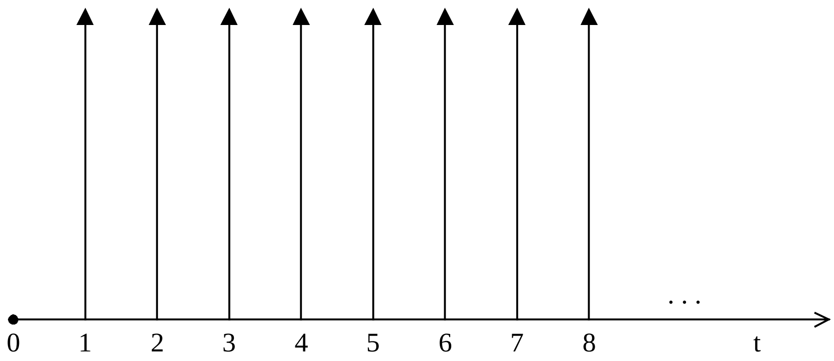


Figure 6 (a)
Cash flow for a buried cable

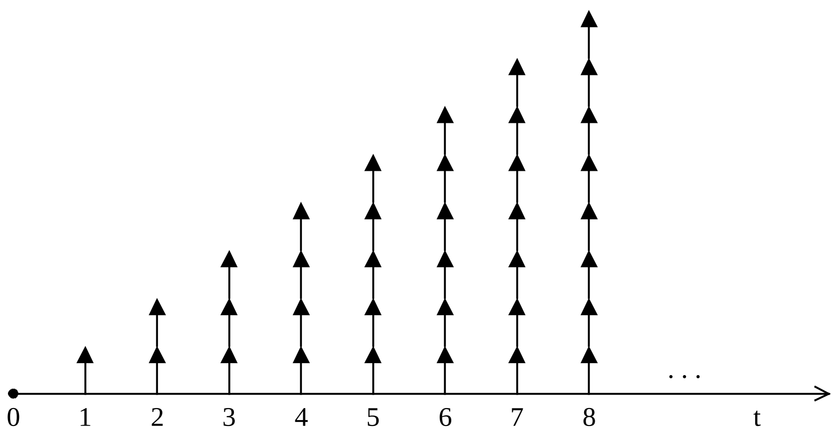


Figure 6 (b)
Cash flow for pgs

The buried cable cash flow is constant over time:

$$a_c = 3500\,MU\,/\,year$$

While for pgs system alternative, the annual charges are a linear function of time:

$$a_p = 1200 \cdot t$$

where t is the time point that the annual charge is considered.

The break-even time is determined when equating the two annual charges.

$$1200t = 3500 \Rightarrow t = 3500 / 1200 = 2.92 \approx 3 years$$

In other words, we save money if we defer for 3 years the placement of the buried cable by using for that period pair gain systems. The present worth of the obtained savings are:

$$PW \quad of \quad savings = (3500 - 1200) / (1+i) + (3500 - 2400) / (1+i)^2 = 2091 + 909 = 3000\,MU$$

Although the whole problem was viewed in a simplified way, its importance is of great value. The appropriate application of pair gain systems in routes, where the existing facilities are used up, can defer the placement of buried cable which calls for considerable initial investments. These investments, in most cases, are not available.

3. Service life

One important parameter that has to be considered in an economic appraisal is the *service life* of the plant. Service life is defined as the period during which the plant exists in the network [1]. When estimating remaining service life of a plant, the following factors should be considered:

- the physical age of the plant;
- the economic life of the plant (e.g. the introduction of new technology may make it economical to replace the existing plant);
- legislation, requirements for better service quality, etc., which necessitate replacement.

The estimation of plant service life is always a forecasting process, particularly where technology forecasts may have a strong influence. Average service lifetime, maintenance plus operating cost as a percentage of the provision cost recommended for network planning purposes are listed in Table 2-1.

No.	Plant	Life time in years (Approximate)	Maintenance + operating cost as a percentage of the cost of provision
1	Poles	25	1,5
2	Wiring	15	1,5
3	Aerial cables	10-20	2,0-5,0
4	Telephone exchange	20	5,0
5	Transmission equipment	20	5,0
6	Power plant	20	5,0
7	Buried cables	40	1,0
8	Cabinets	20	1,5
9	Ducts	60	1,0
10	Tunnels	70	1,0
11	Buildings	60	2,0-5,0

Table 2-1
Average service life, maintenance and operating cost

These figures are intended to give guidance for estimation of plant life for economic studies which should, however, be based as far as possible on the circumstances applicable to the particular case.

4. Present value factor

When making cost calculation, it is sometimes convenient to express the total costs of scheme in terms of its provision costs by including the appropriate additions to cover the costs of replacement, maintenance and operation. This may be achieved by multiplying the first provision costs by the following present worth factor:

$$\mu = 1 + \frac{1-s}{(1+i)^T - 1} + \frac{u}{i} \qquad \textbf{11}$$

where:

T is the service life of a plant
s is the scrap value of retired plant (reduced by the dismantling costs) in relation to provision costs
u is the annual operating plus maintenance costs in relation to provision costs
i is the interest rate (expressed in decimals).

In this expression, the first term is proportional to the provision costs, the second to the net replacement costs (for an infinite period of time) and the third to the maintenance + operating costs (for an infinite period of time).

In most cases, the scrap value of retired plant is almost absorbed by dismantling costs so that $s = 0$.

Assuming s = 0, the present value factor can be rewritten:

$$\mu = \frac{(1+i)^T}{(1+i)^T - 1} + \frac{u}{r} \qquad \textbf{11 a}$$

The full annual charges of a scheme may be obtained from the provision costs by multiplying them by the present value factor and by the interest rate i.

$$a = c \cdot \mu \cdot i$$

c is provision cost
a is annual charge
μ is present value factor
i is interest rate

Example 9:

Calculate the present value factor (pvf) for:

A. Buried cable:

- Service life T = 40 years
- Scrap value is absorbed by dismantling cost s = 0
- Annual operating + maintenance costs = 2 %
- Interest rate = 10 %

We have for (pvf):

$$\mu = 1 + \frac{1-s}{(1+i)^T - 1} + \frac{u}{i} = 1 + \frac{1-0}{(1+0.1)^{40} - 1} + \frac{0.02}{0.1} = 1.223$$

B. PCM transmission system

- Service life T =15 years
- Scrap value s = 0
- Annual operating + maintenance costs = 5 %
- Interest rate = 10 %

We have:

$$\mu = 1 + \frac{1-0}{(1+0.1)^{15}-1} + \frac{0.05}{0.1} = 1.815$$

<u>*Example 10*</u>:

We can provide facilities to subscribers adopting either:

Alternative A

- Service life T = 40 years
- Cost of provision C = 2500 MU
- Maintenance + operating costs u = 2 %
- Scrap value s = 0

or

Alternative B

- Service life T = 15 years
- Cost of provision C = 1800 MU
- Maintenance + operating costs u = 5 %
- Scrap value s = 0
- Interest rate i = 10 %

We want to find which one is the most economical.

Present value factor for alternative A

$$\mu_A = 1 + \frac{1}{(1+0.1)^{40}-1} + \frac{0.02}{0.1} = 1.223$$

Present worth of expenditure for alternative A

$$PW_A = \mu_A c_A = 1.223 \times 2500 = 3057.5\, MU$$

Present value factor for alternative B

$$\mu_B = 1 + \frac{1}{(1+0.1)^{15}-1} + \frac{0.05}{0.1} = 1.815$$

Present worth of expenditure for alternative B

$$PW_B = \mu_B c_B = 1.815 \times 1.800 = 3.267\, MU$$

Alternative A is more economic.

5. References

1. Local network planning, ITU/CCITT
2. General network planning, ITU/CCITT
3. AT&T, Engineering Economy, 3rd Edition, Mc Graw Hill, 1977.